



On the Well-Posedness of the Hall-Magnetohydrodynamics with the Ion-Slip Effect

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Abstract. The existence of global weak solutions is established to the magnetohydrodynamics (MHD) equations with Hall and ion-slip effects in a bounded domain, which coincide with the Hall-MHD equations with and without ion-slip effect for the complementary choices of the parameter $\gamma = 1$ and $\gamma = 0$, respectively. It is also shown that a similar result holds in the whole space. Moreover, the local existence of a unique strong solution and the global well-posedness for small initial data are obtained to the Hall-MHD equations with and without ion-slip effect in both a cubic bounded domain with a flat boundary condition and the whole space. Furthermore, the vanishing viscosity limit to inviscid MHD equations is studied without the ion-slip effect ($\gamma = 0$) in the cubic bounded domain with the flat boundary condition and with the ion-slip effect ($\gamma = 1$) in the whole space, respectively.

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1. Introduction

In recent years the magnetohydrodynamics (MHD) has caught a great deal of attention by physicists and mathematicians due to its physical importance, complexity, rich phenomena, and mathematical challenges. The applications of MHD cover a very broad range of problems in geophysics, astrophysics, cosmology, sensors, engineering, and magnetic drug targeting. For instance, MHD is used to the devices of electromagnetic stirring, nuclear reactors, and plasma confinement. MHD system is a mathematical model which is a combination of the Navier–Stokes equations and Maxwell’s equations, for the low-frequency interaction between electrically conducting fluids and electromagnetic fields. There are many introductory books for MHD [12, 20, 27].

In this paper, we consider the incompressible resistive viscous Hall-MHD equations with or without ion-slip effect in a domain $\Omega \subset \mathbb{R}^3$ for the uniform density of fluid $\rho \equiv 1$:

$$\frac{\partial u}{\partial t} - \mu \Delta u = N_1(\Pi, u, B), \quad (1.1)$$

$$\frac{\partial B}{\partial t} - \eta \Delta B = N_2(\gamma, u, B), \quad (1.2)$$

$$\operatorname{div} u = 0, \quad \operatorname{div} B = 0, \quad (1.3)$$

$$u(0, x) = u^0(x), \quad B(0, x) = B^0(x), \quad (1.4)$$

where $u(t, x)$ is the fluid velocity, $B(t, x)$ is the magnetic field, Π is the scalar pressure of the fluid, μ, η are positive constant coefficients (viscosity, magnetic diffusivity), and

$$N_1(\Pi, u, B) = -(\operatorname{curl} u \times u) + (\operatorname{curl} B \times B) - \nabla \Pi, \quad (1.5)$$

$$N_2(\gamma, u, B) = -\operatorname{curl}(\operatorname{curl} B \times B)$$

$$\begin{aligned}
& -\gamma \operatorname{curl}(B \times (\operatorname{curl} B \times B)) \\
& + \operatorname{curl}(u \times B).
\end{aligned} \tag{1.6}$$

where γ is a nonnegative ion-slip coefficient.

The Hall and ion-slip effect terms are written as $-\operatorname{curl}(\operatorname{curl} B \times B)$ and $-\gamma \operatorname{curl}(B \times (\operatorname{curl} B \times B))$ in (1.6) respectively. These two terms make (1.1)–(1.4) become a quasilinear problem, and make the problem more difficult than general MHD equations. In the present paper, for simplicity, we consider $\gamma = 0$ or 1.

For $\gamma = 0$, it recovers the Hall-MHD equations, which was derived in [2] from the two-fluid isothermal Euler–Maxwell system for the electrons and ions and a kinetic equation for the ion distribution function of the plasma. A important work in [23] initiated a study of the Hall-MHD system. Comparing with the usual MHD equations, we see the appearance of the Hall term related to magnetic reconnection (see [26]). The Hall term $-\operatorname{curl}(\operatorname{curl} B \times B)$ in (1.2) is derived from the generalized Ohm’s law and involves second order derivatives. Its mathematical properties have been studied in several works [3, 6, 7]. In [6, 7], the global existence, local well-posedness, global well-posedness for small initial data, and blow-up criterion were shown in \mathbb{R}^3 . Comparing with the results in [6], regularity conditions of initial data for strong solutions were improved in [3] from $H^m(\mathbb{R}^3)$ with $m > 5/2$ to $H^2(\mathbb{R}^3)$. Moreover, several results on well-posedness of the incompressible Hall-MHD system can be found in [8, 33–36], and on the blow-up criteria, see [15, 17, 21, 33, 36]. For works on the compressible Hall-MHD system, we refer to [16, 18] and references therein. The stochastic Hall-MHD system which is forced by a random noise or random noises is studied in [41, 42]. Authors in [42] show that in the stochastic case, the global well-posedness for small initial data can be obtained even with zero viscous diffusion, highly in contrast to the deterministic case.

For $\gamma \neq 0$, this system appears originally in [31], and was studied as linear initial boundary value problems for the Hall-MHD equations with the ion-slip term. In this case, little mathematical results can be found for Hall-MHD equations with ion-slip effect. The ion-slip effect term $-\gamma \operatorname{curl}(B \times (\operatorname{curl} B \times B))$ is originated from a relative drift ion, electrons, and neutral particles by the electromagnetic field and also obtained from the generalized Ohm’s law.

The origin of the Hall term and ion-slip effect term is well explained in [13]. In a bounded domain with general boundary conditions (1.7) and (3.9) for u and B , and for any given $0 < T < \infty$, the authors in [25] showed that there exists small initial data which depends on T such that a unique solution exists in $0 \leq t \leq T$. The local well-posedness was shown and some regularity criteria were constructed in [14].

Without the Hall and ion-slip terms in (1.1)–(1.4), (i.e., general MHD equations) several results can be found in [10, 19, 28, 29, 37, 40]. Especially, a vanishing viscosity limit in a bounded domain is studied in [4, 38–40] for Navier–Stokes equations and MHD equations.

For a bounded domain Ω , we assume that the domain Ω is surrounded by perfect conducting walls such that there is no surface current on $\partial\Omega$, and we impose $B = 0$ on $\partial\Omega$ as in [9, 22]. In addition, we consider slip and no-slip boundary conditions for u . Combining these boundary conditions for u and B , we will deal with the two different boundary conditions on the bounded domain Ω :

$$\operatorname{curl} u \times \vec{n} = 0, \quad u \cdot \vec{n} = 0 \text{ on } \partial\Omega, \tag{1.7}$$

$$B = 0 \text{ on } \partial\Omega, \tag{1.8}$$

and

$$u = 0 \text{ on } \partial\Omega, \tag{1.9}$$

$$B = 0 \text{ on } \partial\Omega, \tag{1.10}$$

where \vec{n} is the outward unit normal vector to $\partial\Omega$. Our goal in the present paper is to show the existence of global weak solutions for large initial data, local well-posedness, global well-posedness, and vanishing viscosity limit $\mu \rightarrow 0$ in the framework of strong solutions for small initial data in both a bounded domain Ω with the boundary conditions (1.7)–(1.8) and (1.9)–(1.10) and the whole space \mathbb{R}^3 . There are two main difficulties in achieving these goals as described below.

The first main difficulty comes from obtaining the higher-order energy estimates for initial data in $H^n(\Omega)$ for $n \geq 2$, which should be solved in order to establish local well-posedness for our system (1.1)–(1.4). In the case of the whole space \mathbb{R}^3 , to derive the higher-order energy estimates are relatively obtainable compared with the case of a bounded domain since we do not have to care about boundary. In fact, the higher-order energy estimates of Hall-MHD without ion-slip effect in the whole space \mathbb{R}^3 were established in [6]. In the case of resistive viscous MHD (without the Hall and ion-slip terms), the higher-order energy estimates for initial data in $H^1(\Omega)$ can be also obtained in a smooth bounded domain, and it derives the local well-posedness, and global well-posedness for small initial data in $H^1(\Omega)$ (see [29, 40]). However, even in the case of general MHD (without the Hall and ion-slip terms), it is difficult to derive the higher-order energy estimate for initial data in $H^n(\Omega)$ ($2 \leq n \in \mathbb{N}$) because of the compatibility issues of the nonlinear terms. In our case, nonlinear terms (the Hall and ion-slip terms) are quasilinear, so the higher-order energy estimates for small initial data in $H^1(\Omega)$ are insufficient to show the local well-posedness. It means that we need to obtain the higher-order energy estimates for small initial data in $H^n(\Omega)$ for $2 \leq n \in \mathbb{N}$. Hence, we consider suitable functional spaces and a domain with boundary conditions, which is the smooth bounded domain with flat boundary conditions inspired by [40]:

$$\mathcal{C} = \left\{ (x_1, x_2, x_3) : (x_1, x_2) \in [0, 1]_{\text{per}}^2, x_3 \in (0, 1) \right\}, \quad (1.11)$$

with the boundary condition only on two opposite faces:

$$\partial\mathcal{C} = \left\{ (x_1, x_2, x_3) : (x_1, x_2) \in [0, 1]_{\text{per}}^2, x_3 = 0 \text{ or } 1 \right\}, \quad (1.12)$$

where x_1 and x_2 are periodic. Then we will show that the nonlinear terms can match compatibility issues in suitable settings (see Lemma 7), and we show that higher-order energy estimates for initial data in $H^m(\mathcal{C})$ with $m = 2, 3$ can be obtained (see Lemma 9 and Lemma 10). Then the desired higher-order energy estimates will help us to show local well-posedness, global well-posedness for small initial data in $H^2(\mathcal{C})$ and vanishing viscosity limit $\mu \rightarrow 0$ for initial data in $H^3(\mathcal{C})$ without the ion-slip effect ($\gamma = 0$).

The second main difficulty is originated from obtaining finer estimates for the Hall and ion-slip terms on the right-hand side in (1.2) for initial data in $H^2(\Omega)$, which are not easy to be controlled. It is also relatively obtainable in the case of \mathbb{R}^3 compared to the case of a bounded domain. As a matter of fact, in order to obtain finer higher-order estimates for the Hall term, authors in [6] take the differential operator D^α for any $0 \leq \alpha \in \mathbb{Z}$, and use the product rule of the operator D^α , which allows for obtaining the desired results due to the fact:

$$\int_{\mathbb{R}^3} D^\alpha (a \times b) \cdot D^\alpha a dx = \sum_{\substack{n+m=\alpha, \\ 0 \leq n \in \mathbb{Z}, 1 \leq m \in \mathbb{Z}}} \int_{\mathbb{R}^3} (D^n a \times D^m b) \cdot D^\alpha a dx, \quad (1.13)$$

for any $a(x), b(x) \in C_0^\infty(\mathbb{R}^3)^3$. However, in our case, we have additionally the ion-slip term, and we need to take curl^l with $0 \leq l \leq 2$ in order to control the boundary conditions. Unlike the differential operator D^l , curl^l with $0 \leq l \leq 2$ does not satisfy the product rule as like (1.13), i.e.,

$$\text{curl}^l (a \times b) \neq \sum_{\substack{n+m=l, \\ 0 \leq n, m \in \mathbb{Z}}} \text{curl}^n a \times \text{curl}^m b,$$

for any $a(x), b(x) \in C_0^\infty(\mathbb{R}^3)^3$. This point makes it difficult to get the finer estimates for the Hall and ion-slip terms in a bounded domain. To overcome it, we use the flat boundary domain \mathcal{C} and suitable function spaces which we referred to in order to derive the finer estimates for the Hall and ion-slip terms in a bounded domain.

The paper is organized as follows. Preliminary notations and main results will be presented in Sect. 2. Useful embedding lemmas and prior estimates will be introduced in Sect. 3. The existence of global weak solutions for (1.1)–(1.4) with (1.7)–(1.8) by using the Galerkin approximation method (for details, see [29]) is obtained in Sect. 4. The local well-posedness for arbitrary initial data, the global well-posedness for small initial data, and vanishing viscosity limit $\mu \rightarrow 0$ without the ion-slip effect ($\gamma = 0$) in flat

bounded domains will be proved in Sect. 5. Finally, our main results in the whole space \mathbb{R}^3 are discussed in Sect. 6.

2. Main Results

2.1. Notations

We use the differential operator $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $0 \leq \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$, and $0 \leq |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \in \mathbb{Z}$, and $D^m = \sum_{|\alpha| \leq m} D^\alpha$ for $0 \leq m \in \mathbb{Z}$. In addition, we put some useful vector identities which are used throughout the paper. For any three vectors $a, b, c \in \mathbb{R}^3$,

$$-\Delta a = \operatorname{curl}^2 a - \nabla (\operatorname{div} a), \quad (2.1)$$

$$\nabla (a \cdot b) = (a \cdot \nabla) b + (b \cdot \nabla) a + a \times \operatorname{curl} b + b \times \operatorname{curl} a, \quad (2.2)$$

$$\operatorname{div} (a \times b) = b \cdot (\operatorname{curl} a) - a \cdot (\operatorname{curl} b), \quad (2.3)$$

$$\operatorname{curl} (a \times b) = a (\operatorname{div} b) - b (\operatorname{div} a) + (b \cdot \nabla) a - (a \cdot \nabla) b, \quad (2.4)$$

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b). \quad (2.5)$$

Let Ω be a domain in \mathbb{R}^3 . For $0 \leq k \in \mathbb{R}$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ is the Sobolev space belonging to $L^p(\Omega)$. If $p = 2$, we usually write $H^k(\Omega) = W^{k,2}(\Omega)$. For $k \geq 1$, $H_0^k(\Omega)$ is the subspace of $H^k(\Omega)$ consisting of functions vanishing on $\partial\Omega$. We will use $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$ as the norms of $W^{k,p}(\Omega)$ and $H^k(\Omega)$, respectively. We also need to introduce the following spaces:

$$U_0(\Omega) = \{f \in L^2(\Omega) : \operatorname{div} f = 0, f \cdot \vec{n} = 0\},$$

$$U_1(\Omega) = U_0(\Omega) \cap H^1(\Omega),$$

$$U_2(\Omega) = \{f \in U_1(\Omega) \cap H^2(\Omega) : \operatorname{curl} f \times \vec{n} = 0 \text{ on } \partial\Omega\}$$

$$U_k(\Omega) = U_2(\Omega) \cap H^k(\Omega) \text{ for } 3 \leq k \in \mathbb{N},$$

$$V_0(\Omega) = U_0(\Omega), \quad V_1(\Omega) = V_0(\Omega) \cap H_0^1(\Omega),$$

$$V_k(\Omega) = V_1(\Omega) \cap H^k(\Omega) \text{ for } 2 \leq k \in \mathbb{N},$$

and

$$W_0(\Omega) = U_0(\Omega) \times V_0(\Omega),$$

$$W_k(\Omega) = U_k(\Omega) \times V_k(\Omega) \quad \text{for } 1 \leq k \in \mathbb{N},$$

$$Z_0(\Omega) = V_0(\Omega) \times V_0(\Omega),$$

$$Z_k(\Omega) = V_k(\Omega) \times V_k(\Omega) \quad \text{for } 1 \leq k \in \mathbb{N},$$

Moreover, $X(\Omega)^*$ denotes the dual space of any Banach space $X(\Omega)$. The relation $A \lesssim B$ stands for $A \leq CB$, where C denotes a generic constant. Finally, we use the notation $\|(f^1, \dots, f^m)(\cdot)\|_{X(\Omega)}$ as

$$\|(f^1, \dots, f^m)(\cdot)\|_{X(\Omega)}^2 := \sum_{k=1}^m \|f^k(\cdot)\|_{X(\Omega)}^2,$$

for $f^k \in X(\Omega)$ with $k = 1, \dots, m$.

2.2. Main Results

Our goal is to show the existence of global weak solutions for large initial data, local well-posedness, global well-posedness, and vanishing viscosity limit $\mu \rightarrow 0$ in the framework of strong solutions for small initial data in both a bounded domain Ω with the boundary conditions (1.7)–(1.8), (1.9)–(1.10) and the

whole space \mathbb{R}^3 . We give the definitions of a weak solution of (1.1)–(1.4) with two different boundary conditions (1.7)–(1.8) and (1.9)–(1.10).

Definition 1. Let $\gamma = 0$ or 1. Assume that $T > 0$, Ω is a smooth domain, and $(u^0, B^0) \in W_0(\Omega)$ (or $Z_0(\Omega)$) is given. We say that $(u, B) \in L^\infty([0, T]; W_0(\Omega)) \cap L^2([0, T]; W_1(\Omega))$ (or $L^\infty([0, T]; Z_0(\Omega)) \cap L^2([0, T]; Z_1(\Omega))$) with $\frac{\partial u}{\partial t} \in L^{4/3}([0, T]; U_1(\Omega)^*)$ (or $L^{4/3}([0, T]; V_1(\Omega)^*)$), $\frac{\partial B}{\partial t} \in L^{4/3}([0, T]; V_{2+\gamma}(\Omega)^*)$ is a weak solution of (1.1)–(1.4) with (1.7)–(1.8) (or (1.9)–(1.10)) if it satisfies (1.4) and

$$\begin{aligned} & \int_0^t \int_\Omega \left(\frac{\partial u}{\partial t} \cdot v + \mu \operatorname{curl} u \cdot \operatorname{curl} v \right) dx dt \\ &= \int_0^t \int_\Omega ((\operatorname{curl} B \times B) \cdot v - (\operatorname{curl} u \times u) \cdot v) dx dt, \\ & \int_0^t \int_\Omega \left(\frac{\partial B}{\partial t} \cdot w + \eta \operatorname{curl} B \cdot \operatorname{curl} w \right) dx dt \\ &= \int_0^t \int_\Omega ((u \times B) \cdot \operatorname{curl} w - (\operatorname{curl} B \times B) \cdot \operatorname{curl} w) dx dt \\ & \quad - \gamma \int_0^t \int_\Omega (B \times (\operatorname{curl} B \times B)) \cdot \operatorname{curl} w dx dt, \end{aligned}$$

for every $t \in [0, T]$ and any test functions (v, w) such that

$$(v, w) \in C^\infty([0, T]; C_\sigma^\infty(\Omega) \times C_{c,\sigma}^\infty(\Omega)) \quad (\text{or } (v, w) \in C^\infty([0, T]; C_{c,\sigma}^\infty(\Omega) \times C_{c,\sigma}^\infty(\Omega))).$$

where

$$\begin{aligned} C_{c,\sigma}^\infty(\Omega) &= \{f(x) \in C_c^\infty(\Omega) : \operatorname{div} f(x) = 0\}, \\ C_\sigma^\infty(\Omega) &= \{f(x) \in C^\infty(\Omega) : \operatorname{div} f(x) = 0, \quad f(x) \cdot \vec{n}|_{\partial\Omega} = 0\}. \end{aligned}$$

We are now ready to state our main results. We would like to emphasize that we set $\gamma = 0$ or 1 in Theorem 1–Theorem 3, but we only set $\gamma = 0$ in Theorem 4. The first results concern the existence of a global weak solution of (1.1)–(1.4).

Theorem 1. Let Ω be a smooth bounded domain, $T > 0$, and $\gamma = 0$ or 1. The following statements hold.

- (A) If $(u^0, B^0) \in W_0(\Omega)$ (or $Z_0(\Omega)$), there exists a global weak solution (u, B) of (1.1)–(1.4) with (1.7)–(1.8) (or (1.9)–(1.10)) in the sense of Definition 1.
- (B) If $(u^0, B^0) \in Z_0(\mathbb{R}^3)$, there exists a global weak solution (u, B) of (1.1)–(1.4) in the sense of Definition 1.

The next result concerns the local well-posedness of (1.1)–(1.4) on a smooth bounded domain with flat boundary case \mathcal{C} defined in (1.11)–(1.12).

Theorem 2. Let $\gamma = 0$ or 1. The following statements hold.

- (A) If $(u^0, B^0) \in W_2(\mathcal{C})$ (or $Z_2(\mathcal{C})$), there exists a unique local-in-time solution (u, B) of (1.1)–(1.4) with (1.7)–(1.8) (or (1.9)–(1.10)) such that

$$(u, B) \in L^\infty([0, T_0]; W_2(\mathcal{C})) \cap L^2([0, T_0]; W_3(\mathcal{C})) \quad (\text{or } (u, B) \in L^\infty([0, T_0]; Z_2(\mathcal{C})) \cap L^2([0, T_0]; Z_3(\mathcal{C}))),$$
 for some $T_0 > 0$.
- (B) If $(u^0, B^0) \in Z_2(\mathbb{R}^3)$, there exists a unique local-in-time solution (u, B) of (1.1)–(1.4) such that

$$(u, B) \in L^\infty([0, T_0]; Z_2(\mathbb{R}^3)) \cap L^2([0, T_0]; Z_3(\mathbb{R}^3)),$$
 for some $T_0 > 0$.

The next main theorem shows the global well-posedness for (1.1)–(1.4) with small initial data on the smooth domain with the flat boundary case.

Theorem 3. *Let $\gamma = 0$ or 1. The following statements hold.*

- (A) For any initial data $(u^0, B^0) \in W_2(\mathcal{C})$ (or $Z_2(\mathcal{C})$), there exists $\delta > 0$ such that if $\|u^0(\cdot)\|_{H^2(\mathcal{C})} + \|B^0(\cdot)\|_{H^2(\mathcal{C})} < \delta$, there exists a unique global solution (u, B) of (1.1)–(1.4) with (1.7)–(1.8) (or (1.9)–(1.10)) such that
- $$(u, B) \in L^\infty([0, \infty); W_2(\mathcal{C})) \cap L^2([0, \infty); W_3(\mathcal{C})) \quad (\text{or } (u, B) \in L^\infty([0, \infty); Z_2(\mathcal{C})) \cap L^2([0, \infty); Z_3(\mathcal{C}))).$$
- (B) For any initial data $(u^0, B^0) \in Z_2(\mathbb{R}^3)$, there exists $\delta > 0$ such that if $\|u^0(\cdot)\|_{H^2(\mathbb{R}^3)} + \|B^0(\cdot)\|_{H^2(\mathbb{R}^3)} < \delta$, there exists a unique global solution $(u, B) \in L^\infty([0, \infty); Z_2(\mathbb{R}^3)) \cap L^2([0, \infty); Z_3(\mathbb{R}^3))$ of (1.1)–(1.4).

The last our main theorem shows the vanishing viscosity limit $\mu \rightarrow 0$ for (1.1)–(1.4) without the ion-slip effect ($\gamma = 0$).

Theorem 4. *The following statements hold.*

- (A) Let $\gamma = 0$. For any initial data $(u^0, B^0) \in W_3(\mathcal{C})$ (or $Z_3(\mathcal{C})$), there exists $T_0 > 0$ such that solutions $(u_{\mu, \eta}, B_{\mu, \eta})$ of (1.1)–(1.4) for $\mu, \eta > 0$ with (1.7)–(1.8) (or (1.9)–(1.10)) converge to $(u_{0, \eta}, B_{0, \eta})$ as $\mu \rightarrow 0$:

$$(u_{\mu, \eta}, B_{\mu, \eta}) \rightarrow (u_{0, \eta}, B_{0, \eta}) \text{ in } L^q([0, T_0]; H^3(\mathcal{C})), \quad (2.6)$$

$$(u_{\mu, \eta}, B_{\mu, \eta}) \rightarrow (u_{0, \eta}, B_{0, \eta}) \text{ in } C([0, T_0]; H^2(\mathcal{C})), \quad (2.7)$$

for any $q \in [1, \infty)$ and

$$\|u_{\mu, \eta} - u_{0, \eta}\|_{H^2(\mathcal{C})}^2 + \|B_{\mu, \eta} - B_{0, \eta}\|_{H^2(\mathcal{C})}^2 \lesssim C(T_0) \mu, \quad (2.8)$$

for some $C(T_0) > 0$, where $(u_{0, \eta}, B_{0, \eta})$ is a solution of (1.1)–(1.4) for $\mu = 0$ with (1.7)–(1.8) (or (1.9)–(1.10)).

- (B) Let $\gamma = 0$ or 1. For any initial data $(u^0, B^0) \in Z_3(\mathbb{R}^3)$, solutions $(u_{\mu, \eta}, B_{\mu, \eta})$ of (1.1)–(1.4) for $\mu, \eta > 0$ converge to $(u_{0, \eta}, B_{0, \eta})$ as $\mu \rightarrow 0$:

$$(u_{\mu, \eta}, B_{\mu, \eta}) \rightarrow (u_{0, \eta}, B_{0, \eta}) \text{ in } L^q([0, T_0]; H^3(\mathbb{R}^3)),$$

$$(u_{\mu, \eta}, B_{\mu, \eta}) \rightarrow (u_{0, \eta}, B_{0, \eta}) \text{ in } C([0, T_0]; H^2(\mathbb{R}^3)),$$

for any $q \in [1, \infty)$ and

$$\|u_{\mu, \eta} - u_{0, \eta}\|_{H^2(\mathbb{R}^3)}^2 + \|B_{\mu, \eta} - B_{0, \eta}\|_{H^2(\mathbb{R}^3)}^2 \lesssim C(T_0) \mu,$$

for some $C(T_0) > 0$, where $(u_{0, \eta}, B_{0, \eta})$ is a solution of (1.1)–(1.4) for $\mu = 0$.

Remark 1. Indeed, in the case of the whole space \mathbb{R}^3 , Theorem 4 (B) is valid for not only $\gamma = 0$ but also $\gamma = 1$. We will explain the difference of the vanishing viscosity limit results for (1.1)–(1.4) between in the cases \mathcal{C} and \mathbb{R}^3 in Sect. 6.

We skip the proofs of Theorem 1–Theorem 4 in the case of (1.9)–(1.10) since the proofs are similar to the proofs of Theorem 1–Theorem 4 in the case of (1.7)–(1.8).

3. Preliminaries

In this section, we introduce the well-known embedding results without proofs which help us to estimate regularities for (1.1)–(1.4), and basic prior estimates of (1.1)–(1.4) with (1.7)–(1.8) are also presented. We define several function spaces in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ for $1 \leq k \in \mathbb{Z}$:

$$H_n^k(\Omega) = \{f \in H^k(\Omega) : \operatorname{div} f = 0, f \times \vec{n} = 0 \text{ on } \partial\Omega\},$$

$$G_h(\Omega) = \{f = \nabla \varphi : \operatorname{div} f = 0, \varphi = C \text{ on } \partial\Omega, C \in \mathbb{R}\},$$

$$H_{nc}^k(\Omega) = H_n^k(\Omega) \setminus (H_n^k(\Omega) \cap G_h(\Omega)),$$

$$H_\tau^k(\Omega) = \{f \in H^k(\Omega) : \operatorname{div} f = 0, f \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$$

Lemma 1 (Gagliardo–Nirenberg interpolation inequality [1]). *For any real-valued function $f(x)$ on Ω , it holds*

$$\|D^j f(\cdot)\|_{L^p(\Omega)} \lesssim \left(\|D^m f(\cdot)\|_{L^r(\Omega)}^a \|f(\cdot)\|_{L^q(\Omega)}^{1-a} + \|f(\cdot)\|_{L^s(\Omega)} \right) \text{ for any } s > 0,$$

if the number $a \in \mathbb{R}$ and $j \in \mathbb{N}$ satisfy

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n} \right) a + \frac{1-a}{q},$$

for fixed $1 \leq q, r \leq \infty$, and $m \in \mathbb{N}$, where $\frac{j}{m} \leq a \leq 1$.

The detailed proofs of the next two lemmas are given in [5, 11].

Lemma 2. *Let $1 \leq s \in \mathbb{N}$. For any $f(x) \in H^s(\Omega)$, it holds*

$$\begin{aligned} \|f(\cdot)\|_{H^s(\Omega)} &\lesssim \|\operatorname{curl} f(\cdot)\|_{H^{s-1}(\Omega)} + \|\operatorname{div} f(\cdot)\|_{H^{s-1}(\Omega)} \\ &\quad + \|\vec{n} \cdot f(\cdot)\|_{H^{s-1/2}(\partial\Omega)} + \|f(\cdot)\|_{H^{s-1}(\Omega)}. \end{aligned}$$

Epecially, if $f \in H_\tau^1(\Omega)$,

$$\|f(\cdot)\|_{H^1(\Omega)} \lesssim \|\operatorname{curl} f(\cdot)\|_{L^2(\Omega)}.$$

Lemma 3. *Let $1 \leq s \in \mathbb{N}$. For any $f(x) \in H^s(\Omega)$, it holds*

$$\begin{aligned} \|f(\cdot)\|_{H^s(\Omega)} &\lesssim \|\operatorname{curl} f(\cdot)\|_{H^{s-1}(\Omega)} + \|\operatorname{div} f(\cdot)\|_{H^{s-1}(\Omega)} \\ &\quad + \|\vec{n} \times f(\cdot)\|_{H^{s-1/2}(\partial\Omega)} + \|f(\cdot)\|_{H^{s-1}(\Omega)}. \end{aligned}$$

Moreover, if $f(x) \in H_{nc}^1(\Omega)$,

$$\|f(\cdot)\|_{L^2(\Omega)} \lesssim \|\operatorname{curl} f(\cdot)\|_{L^2(\Omega)}.$$

To deal with the nonlinear terms in the proof of existence of the weak solutions, we need the following lemma.

Lemma 4 (Aubin–Lions Lemma [30]). *Let X_0 , X_1 , and X_2 be Banach spaces such that*

$$X_0 \subset X_1 \subset X_2,$$

where X_0 is compactly embedded in X_1 and X_1 is continuously embedded in X_2 . Then for any $1 \leq p, q \leq \infty$

$$\left\{ f(t, x) \in L^p([0, T]; X_0) : \frac{\partial f(t, x)}{\partial t} \in L^q([0, T]; X_2) \right\}$$

is compactly embedded in $L^p([0, T]; X_1)$ if $p < \infty$, and $C([0, T]; X_1)$ if $p = \infty$, $q > 1$.

The last lemma is related to weak convergence.

Lemma 5. *Let Ω be a domain in \mathbb{R}^3 . Define sequences $\{a_k\}_{k=1}^\infty$, $\{b_k\}_{k=1}^\infty$ in $L^p(\Omega)$ and $L^q(\Omega)$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If*

$$\begin{aligned} a_k &\rightarrow a \text{ weakly in } L^p(\Omega), \\ b_k &\rightarrow b \text{ strongly in } L^q(\Omega), \end{aligned}$$

then

$$a_k b_k \rightarrow ab \text{ weakly in } L^1(\Omega). \quad (3.1)$$

Proof. Notice $a_k b_k - ab = a_k (b_k - b) + (a_k - a) b$. For any $\phi \in L^\infty(\Omega)$, we have

$$\int_{\Omega} (a_k b_k - ab) \phi dx = \int_{\Omega} a_k (b_k - b) \phi dx + \int_{\Omega} (a_k - a) b \phi dx.$$

With the assumptions, we obtain

$$\begin{aligned} \int_{\Omega} a_k (b_k - b) \phi dx &\leq \|\phi(\cdot)\|_{L^\infty(\Omega)} \|a_k(\cdot)\|_{L^p(\Omega)} \|b_k - b\|_{L^q(\Omega)} \\ &\leq C \|\phi(\cdot)\|_{L^\infty(\Omega)} \|b_k - b\|_{L^q(\Omega)}, \\ \int_{\Omega} (a_k - a) b \phi dx &\rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

for some $C > 0$, which implies (3.1). \square

We now establish a-priori basic estimates for Hall-MHD equations (1.1)–(1.4) with (1.7)–(1.8). By multiplying both sides of equation (1.1) by u and by integrating in Ω , we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\Omega)}^2 + \mu \|\operatorname{curl} u(t, \cdot)\|_{L^2(\Omega)}^2 \\ &= - \int_{\Omega} (\operatorname{curl} u \times u) \cdot u dx + \int_{\Omega} (\operatorname{curl} B \times B) \cdot u dx + \int_{\Omega} \Pi (\operatorname{div} u) dx \\ &\quad - \int_{\partial\Omega} \vec{n} \cdot (\Pi u) ds - \int_{\partial\Omega} \vec{n} \cdot (\operatorname{curl} u \times u) ds. \end{aligned} \quad (3.2)$$

In order to estimate the first term on the right hand side of (3.2) we use the basic vector calculus identity:

$$- \int_{\Omega} (\operatorname{curl} u \times u) \cdot u dx = - \int_{\Omega} (\operatorname{curl} u) \cdot (u \times u) dx = 0.$$

Using the identity

$$\operatorname{curl} B \times B = B \cdot \nabla B - \frac{1}{2} \nabla |B|^2,$$

and the boundary conditions (1.7)–(1.8), the second term on the right-hand side of (3.2) can be written as follows:

$$\int_{\Omega} (\operatorname{curl} B \times B) \cdot u dx = - \int_{\Omega} \left((B \cdot \nabla u) \cdot B + \frac{1}{2} \nabla (|B|^2) \cdot u \right) dx.$$

Since $\operatorname{div} u = 0$ on Ω , it is obvious that

$$\int_{\Omega} \Pi (\operatorname{div} u) dx = 0.$$

Next we follow similar argument with B instead of u . Multiplying (1.2) by B yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|B(t, \cdot)\|_{L^2(\Omega)}^2 + \eta \|\operatorname{curl} B(t, \cdot)\|_{L^2(\Omega)}^2 \\ &= \int_{\Omega} \operatorname{curl} (u \times B) \cdot B dx - \int_{\Omega} \operatorname{curl} (\operatorname{curl} B \times B) \cdot B dx \\ &\quad - \gamma \int_{\Omega} \operatorname{curl} (B \times (\operatorname{curl} B \times B)) \cdot B dx - \eta \int_{\partial\Omega} \vec{n} \cdot (\operatorname{curl} B \times B) ds. \end{aligned} \quad (3.3)$$

The first term on the right-hand side of (3.3) can be written as (see (2.4) in [22]):

$$\int_{\Omega} \operatorname{curl} (u \times B) \cdot B dx = \int_{\Omega} \left((B \cdot \nabla u) \cdot B + \frac{1}{2} \nabla (|B|^2) \cdot u \right) dx. \quad (3.4)$$

With the help of the identity

$$\begin{aligned} \operatorname{div}((\operatorname{curl} B \times B) \times B) &= \operatorname{curl}(\operatorname{curl} B \times B) \cdot B \\ &\quad - (\operatorname{curl} B \times B) \cdot (\operatorname{curl} B), \end{aligned} \quad (3.5)$$

the second term on the right-hand side of (3.3) can be written as

$$\begin{aligned} - \int_{\Omega} \operatorname{curl}(\operatorname{curl} B \times B) \cdot B dx &= - \int_{\Omega} (\operatorname{curl} B \times B) \cdot \operatorname{curl} B dx \\ &\quad - \int_{\partial\Omega} \vec{n} \cdot ((\operatorname{curl} B \times B) \times B) ds. \end{aligned} \quad (3.6)$$

Finally, the third term on the right-hand side of (3.3) can be written as

$$\begin{aligned} -\gamma \int_{\Omega} \operatorname{curl}(B \times (\operatorname{curl} B \times B)) \cdot B dx &= -\gamma \int_{\Omega} |(\operatorname{curl} B \times B)|^2 dx \\ &\quad - \gamma \int_{\partial\Omega} \vec{n} \cdot ((B \times (\operatorname{curl} B \times B)) \times B) ds. \end{aligned} \quad (3.7)$$

We then obtain from the relations (3.2)–(3.7) that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\Omega)}^2 + \mu \|\operatorname{curl} u(t, \cdot)\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{2} \frac{d}{dt} \|B(t, \cdot)\|_{L^2(\Omega)}^2 + \eta \|\operatorname{curl} B(t, \cdot)\|_{L^2(\Omega)}^2 + \gamma \|\operatorname{curl} B(t, \cdot) \times B(t, \cdot)\|_{L^2(\Omega)}^2 \\ &= - \int_{\partial\Omega} \vec{n} \cdot (\Pi u) ds - \int_{\partial\Omega} \vec{n} \cdot (\operatorname{curl} u \times u) ds \\ &\quad - \eta \int_{\partial\Omega} \vec{n} \cdot (\operatorname{curl} B \times B) ds - \int_{\partial\Omega} \vec{n} \cdot ((\operatorname{curl} B \times B) \times B) ds \\ &\quad - \gamma \int_{\partial\Omega} \vec{n} \cdot ((B \times (\operatorname{curl} B \times B)) \times B) ds. \end{aligned} \quad (3.8)$$

Remark 2. The estimates obtained in (3.8) are not sufficient to obtain our results in a general domain Ω . Hence, we need to confine ourselves to the boundary conditions. Indeed, if we set

$$\begin{aligned} \operatorname{curl} u \times \vec{n} &= 0, \quad u \cdot \vec{n} = 0 \text{ on } \partial\Omega, \\ B &= 0 \text{ on } \partial\Omega, \end{aligned}$$

then the boundary integrals in equation (3.8) vanish. It is worth mentioning that the physically relevant boundary condition (nonlinear perfectly conducting wall boundary condition) [25, 31]:

$$\begin{aligned} B \cdot \vec{n} &= 0 \text{ on } \partial\Omega, \\ (\operatorname{curl} B + \eta(\operatorname{curl} B \times B) + \gamma(B \times (\operatorname{curl} B \times B))) \times \vec{n} &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (3.9)$$

can be used to remove the boundary integral term in (3.8) due to the fact

$$\begin{aligned} \vec{n} \cdot ((\operatorname{curl} B \times B) \times B) &= B \cdot ((\operatorname{curl} B \times B) \times \vec{n}), \\ \vec{n} \cdot (\operatorname{curl} B \times B) &= B \cdot (\operatorname{curl} B \times \vec{n}), \\ \vec{n} \cdot ((B \times (\operatorname{curl} B \times B)) \times B) &= B \cdot ((B \times (\operatorname{curl} B \times B)) \times \vec{n}). \end{aligned}$$

4. The Proof of Theorem 1 (A)

In this section, our goal is to establish the existence of global weak solutions for equations (1.1)–(1.4) with (1.7)–(1.8). We will use the Galerkin approximate method to obtain the existence of global weak solutions by deriving suitable energy estimates. We then show that the approximated solutions converge to a solution of (1.1)–(1.4). There exists a complete orthogonal basis $\{\kappa_k = (v_k, \beta_k)\}_{k=1}^{\infty} \subset W_2(\Omega)$ of

$W_0(\Omega)$ which are eigenvalues of the stokes operator $-\Delta$, with eigenvalues $\{\lambda_k = (\lambda_{1,k}, \lambda_{2,k})\}_{k=1}^\infty$ such that $0 < \lambda_{1,k} \rightarrow \infty$, and $0 < \lambda_{2,k} \rightarrow \infty$ as $k \rightarrow \infty$. There exists a projection $: W_0(\Omega) \rightarrow W_0^N(\Omega)$ with $1 \leq N \in \mathbb{N}$, where $W_0^N(\Omega)$ is the space spanned by $\{\kappa_k\}_{k=1}^N$. We also use the notations $U_0^N(\Omega)$ and $V_0^N(\Omega)$ spanned by $\{v_k\}_{k=1}^N$ and $\{\beta_k\}_{k=1}^N$. Let

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t) v_k(x) \quad \text{and} \quad B(t, x) = \sum_{k=1}^{\infty} B_k(t) \beta_k(x).$$

Let us define a sequence of approximate functions

$$u^N(t, x) := \sum_{k=1}^N u_k(t) v_k(x) \in H^1([0, T]; U_0^N(\Omega)),$$

$$B^N(t, x) := \sum_{k=1}^N B_k(t) \beta_k(x) \in H^1([0, T]; V_0^N(\Omega)),$$

and $P^N = (P_U^N \times P_V^N) : W_0(\Omega) \rightarrow W_0^N(\Omega)$ with $1 \leq N \in \mathbb{N}$ is a natural projection generated by the orthogonal complete basis $\{\kappa_k\}_{k=1}^N$. Notice that since $\partial\Omega$ is C^∞ , by the theory of elliptic operators, each κ_k is a C^∞ -function. We now give the definition of the approximate solutions for a Galerkin system.

Definition 2. Let $\gamma = 0$ or 1 , $T > 0$ and $((u^0)^N(x), (B^0)^N(x)) \in W_0^N(\Omega) \subset C^\infty(\Omega) \times C^\infty(\Omega)$ with $1 \leq N \in \mathbb{N}$ be given. Then $(u^N, B^N) \in H^1([0, T]; W_0^N(\Omega))$ is an approximated solution of (1.1)–(1.4) with (1.7)–(1.8) if it holds

$$\frac{\partial u^N}{\partial t} - \mu \Delta u^N = P_U^N N_1(\Pi, u^N, B^N), \quad (4.1)$$

$$\frac{\partial B^N}{\partial t} - \eta \Delta B^N = P_V^N N_2(\gamma, u^N, B^N), \quad (4.2)$$

where both N_1 and N_2 are defined in (1.5) and (1.6).

With this definition and classical ODE theory, we prove the following lemma.

Lemma 6. Let $\gamma = 0$ or 1 , $T > 0$ and $((u^0)^N, (B^0)^N) \in W_0^N(\Omega) \subset C^\infty(\Omega) \times C^\infty(\Omega)$ with $1 \leq N \in \mathbb{N}$ be given. Then an approximated solution $(u^N, B^N) \in H^1([0, T]; W_0^N(\Omega))$ of (4.1)–(4.12) with $u^N(0, x) = (u^0)^N(x)$ and $B^N(0, x) = (B^0)^N(x)$ satisfies

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u^N(t, \cdot)\|_{L^2(\Omega)}^2 + \mu \|\operatorname{curl} u^N(t, \cdot)\|_{L^2(\Omega)}^2 \\ & + \frac{1}{2} \frac{d}{dt} \|B^N(t, \cdot)\|_{L^2(\Omega)}^2 + \eta \|\operatorname{curl} B^N(t, \cdot)\|_{L^2(\Omega)}^2 \\ & + \gamma \|(\operatorname{curl} B^N \times B^N)(t, \cdot)\|_{L^2(\Omega)}^2 \\ & \leq 0. \end{aligned} \quad (4.3)$$

Hence, for $0 < t \leq T$, it holds

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|(u^N(\tau, \cdot), B^N(\tau, \cdot))\|_{L^2(\Omega)}^2 \\ & + 2 \int_0^t \left\| \left(\mu^{1/2} \operatorname{curl} u^N(\tau, \cdot), \eta^{1/2} \operatorname{curl} B^N(\tau, \cdot) \right) \right\|_{L^2(\Omega)}^2 d\tau \\ & + 2\gamma \int_0^t \|(\operatorname{curl} B^N \times B^N)(\tau, \cdot)\|_{L^2(\Omega)}^2 d\tau \\ & \leq \|u^N(0, \cdot)\|_{L^2(\Omega)}^2 + \|B^N(0, \cdot)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.4)$$

Proof. Clearly, it can be shown that there exists a unique local-in-time solution of (4.1)–(4.12) on $[0, T_N]$ for some $T_N > 0$ by a classical ODE theory. By multiplying v^N, w^N to (4.1)–(4.12) respectively, it satisfies for every $t \in [0, T_N]$ and any $(v^N, w^N) \in C^\infty([0, T_N]; W_0^N(\Omega))$, that

$$\int_{\Omega} \left(\frac{\partial u^N}{\partial t} \cdot v^N + \mu \operatorname{curl} u^N \cdot \operatorname{curl} v^N \right) dx \quad (4.5)$$

$$\begin{aligned} &= \int_{\Omega} \left(-(\operatorname{curl} u^N \times u^N) \cdot v^N + (\operatorname{curl} B^N \times B^N) \cdot v^N \right) dx, \\ &\int_{\Omega} \left(\frac{\partial B^N}{\partial t} \cdot w^N + \eta \operatorname{curl} B^N \cdot \operatorname{curl} w^N + \gamma (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl} w^N \right) dx \\ &= \int_{\Omega} \left((u^N \times B^N) \cdot \operatorname{curl} w^N - (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl} w^N \right) dx, \end{aligned} \quad (4.6)$$

with $u^N(0, x) = (u^0)^N(x)$, $B^N(0, x) = (B^0)^N(x)$. Set $v^N = u^N, w^N = B^N$ in (4.5) and (4.6) for $t \in [0, T_N]$. Then using the relations (3.2)–(3.7) with respect to u^N, B^N , we derive the uniform energy estimate (4.3). Integrating (4.3) in time, we easily obtain (4.4) for $t \in [0, T_N]$, which implies that we can extend T_N to an arbitrary T . \square

We are ready to prove Theorem 1 (A) with the boundary conditions (1.7)–(1.8) by applying Lemma 1–Lemma 6.

Proof of Theorem 1 (A). We need to estimate $\frac{\partial u^N}{\partial t}$ and $\frac{\partial B^N}{\partial t}$ in order to satisfy the hypotheses of Lemma 4. Clearly, for any $(v^N(t, x), w^N(t, x)) \in C^\infty([0, T]; W_0^N(\Omega))$,

$$\begin{aligned} \left| \int_{\Omega} \operatorname{curl} u^N \cdot \operatorname{curl} v^N dx \right| &\leq \|Du^N(t, \cdot)\|_{L^2(\Omega)} \|v^N(t, \cdot)\|_{H^1(\Omega)}, \\ \left| \int_{\Omega} \operatorname{curl} B^N \cdot \operatorname{curl} w^N dx \right| &\leq \|DB^N(t, \cdot)\|_{L^2(\Omega)} \|w^N(t, \cdot)\|_{H^1(\Omega)}. \end{aligned} \quad (4.7)$$

From Lemma 1.3 of Ch. II in [32], we know that

$$\begin{aligned} \left| \int_{\Omega} (\operatorname{curl} u^N \times u^N) \cdot v^N dx \right| &= \left| \int_{\Omega} ((u^N \cdot \nabla) v^N \cdot u^N) dx \right|, \\ \left| \int_{\Omega} (\operatorname{curl} B^N \times B^N) \cdot v^N dx \right| &= \left| \int_{\Omega} ((B^N \cdot \nabla) v^N \cdot B^N) dx \right|. \end{aligned}$$

Based on the Lemma 1–Lemma 3, we obtain the following estimates:

$$\begin{aligned} \left| \int_{\Omega} (\operatorname{curl} u^N \times u^N) \cdot v^N dx \right| &\lesssim \|u^N(t, \cdot)\|_{L^4(\Omega)}^2 \|v^N(t, \cdot)\|_{H^1(\Omega)} \\ &\lesssim \|Du^N(t, \cdot)\|_{L^2(\Omega)}^{3/2} \|u^N(t, \cdot)\|_{L^2(\Omega)}^{1/2} \|v^N(t, \cdot)\|_{H^1(\Omega)}, \\ \left| \int_{\Omega} (\operatorname{curl} B^N \times B^N) \cdot v^N dx \right| &\lesssim \|B^N(t, \cdot)\|_{L^4(\Omega)}^2 \|v^N(t, \cdot)\|_{H^1(\Omega)} \\ &\lesssim \|DB^N(t, \cdot)\|_{L^2(\Omega)}^{3/2} \|B(t, \cdot)\|_{L^2(\Omega)}^{1/2} \|v^N(t, \cdot)\|_{H^1(\Omega)}, \end{aligned} \quad (4.8)$$

uniformly in N . Moreover, by integration by parts, we have

$$\begin{aligned} &\left| \int_{\Omega} (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl} w^N dx \right| \\ &= \left| \int_{\Omega} \left((B^N \cdot \nabla B^N) - \nabla \left(\frac{|B^N|^2}{2} \right) \right) \cdot \operatorname{curl} w^N dx \right| \\ &= \left| \int_{\Omega} B^N \cdot (B^N \cdot \nabla) \operatorname{curl} w^N dx \right|, \end{aligned}$$

which yields

$$\begin{aligned}
 & \left| \int_{\Omega} (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl} w^N dx \right| \\
 & \lesssim \|B^N(t, \cdot)\|_{L^4(\Omega)}^2 \|w^N(t, \cdot)\|_{H^2(\Omega)} \\
 & \lesssim \|DB^N(t, \cdot)\|_{L^2(\Omega)}^{3/2} \|B(t, \cdot)\|_{L^2(\Omega)}^{1/2} \|w^N(t, \cdot)\|_{H^2(\Omega)},
 \end{aligned} \tag{4.9}$$

uniformly in N . We also get

$$\left| \int_{\Omega} (u^N \times B^N) \cdot \operatorname{curl} w^N dx \right| \tag{4.10}$$

$$\begin{aligned}
 & \lesssim \|u^N(t, \cdot)\|_{L^4(\Omega)} \|B^N(t, \cdot)\|_{L^4(\Omega)} \|w^N(t, \cdot)\|_{H^1(\Omega)} \\
 & \lesssim \|Du^N(t, \cdot)\|_{L^2(\Omega)}^{3/4} \|u^N(t, \cdot)\|_{L^2(\Omega)}^{1/4} \|DB^N(t, \cdot)\|_{L^2(\Omega)}^{3/4} \|B^N(t, \cdot)\|_{L^2(\Omega)}^{1/4} \|w^N(t, \cdot)\|_{H^1(\Omega)}
 \end{aligned}$$

$$\begin{aligned}
 & \left| \int_{\Omega} (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl} w^N dx \right| \\
 & \lesssim \|(B^N \times \operatorname{curl} B^N)(t, \cdot)\|_{L^2(\Omega)} \|B^N(t, \cdot)\|_{L^2(\Omega)} \|w^N(t, \cdot)\|_{H^3(\Omega)},
 \end{aligned} \tag{4.11}$$

uniformly in N . We deduce from (4.7)–(4.11)

$$\begin{aligned}
 & \frac{\partial u^N}{\partial t} \rightarrow \frac{\partial u}{\partial t} \text{ weakly in } L^{4/3}([0, T]; U_1(\Omega)^*) , \\
 & \frac{\partial B^N}{\partial t} \rightarrow \frac{\partial B}{\partial t} \text{ weakly in } L^{4/3}([0, T]; V_3(\Omega)^*) \text{ for } \gamma = 1, \\
 & \frac{\partial B^N}{\partial t} \rightarrow \frac{\partial B}{\partial t} \text{ weakly in } L^{4/3}([0, T]; V_2(\Omega)^*) \text{ for } \gamma = 0.
 \end{aligned} \tag{4.12}$$

Using Lemma 4 and the fact

$$U_1(\Omega) \subset\subset U_0(\Omega) \subset U_1(\Omega)^*, \quad V_1(\Omega) \subset\subset V_0(\Omega) \subset V_2(\Omega)^* \subset V_3(\Omega)^*,$$

where $A \subset\subset B$ means that A is compactly embedded in B , we can see that the embeddings of

$$\begin{aligned}
 & \left\{ f \in L^2([0, T]; V_1(\Omega)) : \frac{\partial f}{\partial t} \in L^{4/3}([0, T]; V_2(\Omega)^*) \right\} \longrightarrow L^2([0, T]; V_0(\Omega)), \\
 & \left\{ f \in L^2([0, T]; V_1(\Omega)) : \frac{\partial f}{\partial t} \in L^{4/3}([0, T]; V_3(\Omega)^*) \right\} \longrightarrow L^2([0, T]; V_0(\Omega)),
 \end{aligned}$$

and

$$\left\{ f \in L^2([0, T]; U_1(\Omega)) : \frac{\partial f}{\partial t} \in L^{4/3}([0, T]; U_1(\Omega)^*) \right\} \longrightarrow L^2([0, T]; U_0(\Omega))$$

are compact. Hence, there exists a subsequence $\{u^m, B^m\}_{m=1}^{\infty}$ of $\{u^N, B^N\}_{N=1}^{\infty}$ satisfying (4.12) and

$$(u^m, B^m) \rightarrow (u, B) \text{ strongly in } L^2([0, T]; W_0(\Omega)). \tag{4.13}$$

We easily see that for fixed but arbitrary $T > 0$,

$$\begin{aligned}
 & u^N(t, \cdot) \rightarrow u(t, \cdot) \text{ weakly-star in } L^{\infty}([0, T]; U_0(\Omega)), \\
 & B^N(t, \cdot) \rightarrow B(t, \cdot) \text{ weakly-star in } L^{\infty}([0, T]; V_0(\Omega)), \\
 & u^N(t, \cdot) \rightarrow u(t, \cdot) \text{ weakly in } L^2([0, T]; U_1(\Omega)), \\
 & B^N(t, \cdot) \rightarrow B(t, \cdot) \text{ weakly in } L^2([0, T]; V_1(\Omega)),
 \end{aligned} \tag{4.14}$$

by (4.4) in Lemma 6. Therefore, since the product of weak-strong converging sequences converges weakly (see Lemma 5), applying (4.13)–(4.14), we can show the following:

$$\begin{aligned} \int_0^T \int_{\Omega} ((\operatorname{curl} u^N \times u^N) \cdot v) \, dxdt &\rightarrow \int_0^T \int_{\Omega} ((\operatorname{curl} u \times u) \cdot v) \, dxdt, \\ \int_0^T \int_{\Omega} \operatorname{curl} u^N \cdot \operatorname{curl} v \, dxdt &\rightarrow \int_0^T \int_{\Omega} \operatorname{curl} u \cdot \operatorname{curl} v \, dxdt, \\ \int_0^T \int_{\Omega} (\operatorname{curl} B^N \times B^N) \cdot v \, dxdt &\rightarrow \int_0^T \int_{\Omega} (\operatorname{curl} B \times B) \cdot v \, dxdt, \\ \int_0^T \int_{\Omega} \operatorname{curl} B^N \cdot \operatorname{curl} w \, dxdt &\rightarrow \int_0^T \int_{\Omega} \operatorname{curl} B \cdot \operatorname{curl} w \, dxdt, \end{aligned}$$

and

$$\begin{aligned} \int_0^T \int_{\Omega} (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl} w \, dxdt &\rightarrow \int_0^T \int_{\Omega} (\operatorname{curl} B \times B) \cdot \operatorname{curl} w \, dxdt, \\ \int_0^T \int_{\Omega} (u^N \times B^N) \cdot \operatorname{curl} w \, dxdt &\rightarrow \int_0^T \int_{\Omega} (u \times B) \cdot \operatorname{curl} w \, dxdt, \\ \int_0^T \int_{\Omega} (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl} w \, dxdt &\rightarrow \int_0^T \int_{\Omega} (B \times (\operatorname{curl} B \times B)) \cdot \operatorname{curl} w \, dxdt, \end{aligned}$$

for any $(v(t, x), w(t, x)) \in C^\infty([0, T]; C^\infty_\sigma(\Omega) \times C^\infty_{c,\sigma}(\Omega))$. This completes the proof of Theorem 1 (A). \square

5. The proof of Theorem 2 (A)–Theorem 4 (A)

In this section, we will prove successively Theorem 2 (A), Theorem 3 (A), and Theorem 4 (A). Before the proofs, we first derive some higher-order energy estimates for equations (1.1)–(1.4) in the flat boundary domain \mathcal{C} with (1.7)–(1.8). We first discuss the compatibility issues for nonlinear terms which will be used to control the boundary integral terms associated with the flat boundary condition.

Lemma 7. *For any $(f, g) \in W_2(\mathcal{C}) \cap C^\infty(\mathcal{C})$,*

$$N_1(\Pi, f, g) \in U_2(\mathcal{C}) \cap C^\infty(\mathcal{C}), \quad (5.1)$$

$$N_2(\gamma, f, g) \in V_2(\mathcal{C}) \cap C^\infty(\mathcal{C}), \quad (5.2)$$

where N_1 and N_2 are defined in (1.5) and (1.6).

Proof. First, we consider $N_1(\Pi, f, g)$. Thanks to Proposition 2.5 and 2.6 in [40], it suffices to show

$$\operatorname{curl}(\operatorname{curl} g \times g) \times \vec{n} = 0. \quad (5.3)$$

Using the vector identity and $\operatorname{div} g = 0$, we have

$$\operatorname{curl}(\operatorname{curl} g \times g) = (g \cdot \nabla) \operatorname{curl} g - (\operatorname{curl} g \cdot \nabla) g. \quad (5.4)$$

Clearly, we obtain

$$((g \cdot \nabla) \operatorname{curl} g) \times \vec{n} = 0,$$

owing to

$$g = 0 \text{ on } \partial\mathcal{C}.$$

By the flat boundary condition, for $g = (g_1, g_2, g_3)$, we can derive

$$\begin{aligned}
\operatorname{curl} g &= -\left(\frac{\partial g_2}{\partial x_3} - \frac{\partial g_3}{\partial x_2}, \frac{\partial g_3}{\partial x_1} - \frac{\partial g_1}{\partial x_3}, \frac{\partial g_1}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\right) \\
&= -\left(\frac{\partial g_2}{\partial x_3}, -\frac{\partial g_1}{\partial x_3}, 0\right), \\
\nabla g &= (\nabla g_1, \nabla g_2, \nabla g_3).
\end{aligned} \tag{5.5}$$

Then we also get $(\operatorname{curl} g \cdot \nabla) g \times \vec{n} = 0$ due to the fact

$$\begin{aligned}
&(\operatorname{curl} g \cdot \nabla) g \\
&= -\left(\frac{\partial g_2}{\partial x_3} \frac{\partial g_1}{\partial x_1} - \frac{\partial g_1}{\partial x_3} \frac{\partial g_2}{\partial x_2}, \frac{\partial g_2}{\partial x_3} \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_3} \frac{\partial g_2}{\partial x_2}, \frac{\partial g_2}{\partial x_3} \frac{\partial g_3}{\partial x_1} - \frac{\partial g_1}{\partial x_3} \frac{\partial g_3}{\partial x_2}\right) \\
&= 0.
\end{aligned} \tag{5.6}$$

Therefore, (5.3) can be established. Next we will show (5.2). By Proposition 2.5 and 2.6 in [40], it is enough to show

$$\operatorname{curl}(\operatorname{curl} g \times g) = 0 \text{ on } \partial\mathcal{C}, \tag{5.7}$$

$$\operatorname{curl}(g \times (\operatorname{curl} g \times g)) = 0 \text{ on } \partial\mathcal{C}. \tag{5.8}$$

In fact, it is obvious that (5.7) holds by using (5.4) and (5.6). Moreover, owing to $g = 0$ on $\partial\mathcal{C}$, we obtain

$$\begin{aligned}
\operatorname{curl}(g \times (\operatorname{curl} g \times g)) &= g(\operatorname{div}(\operatorname{curl} g \times g)) - (\operatorname{curl} g \times g)(\operatorname{div} g) \\
&\quad + ((\operatorname{curl} g \times g) \cdot \nabla) g - (g \cdot \nabla)(\operatorname{curl} g \times g) \\
&= 0 \text{ on } \partial\mathcal{C},
\end{aligned}$$

which implies (5.8) holds. This completes the proof of Lemma 7. \square

5.1. Higher-Order Energy Estimates

Using (2.1), (2.3) and noting that for $k \leq N$,

$$P_U^N(-\Delta u^k) = P_U^N(\lambda_{1,k} u^k) = \lambda_{1,k} u^k = -\Delta u^k. \tag{5.9}$$

By operating curl to (4.1) and (4.12), taking inner product with $\operatorname{curl} u^N$ and $\operatorname{curl} B^N$, respectively, and integrating it on \mathcal{C} , we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\operatorname{curl} u^N|^2 dx + \mu \int_{\mathcal{C}} |\operatorname{curl}^2 u^N|^2 dx \\
&= \int_{\mathcal{C}} \operatorname{curl} P_U^N N_1(\Pi, u^N, B^N) \cdot \operatorname{curl} u^N dx \\
&= \int_{\mathcal{C}} \operatorname{curl} N_1(\Pi, u^N, B^N) \cdot \operatorname{curl} u^N dx
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\operatorname{curl} B^N|^2 dx + \eta \int_{\mathcal{C}} |\operatorname{curl}^2 B^N|^2 dx \\
&= \int_{\mathcal{C}} \operatorname{curl} P_V^N N_2(\gamma, u^N, B^N) \cdot \operatorname{curl} B^N dx \\
&= \int_{\mathcal{C}} \operatorname{curl} N_2(\gamma, u^N, B^N) \cdot \operatorname{curl} B^N dx.
\end{aligned} \tag{5.11}$$

Here we have used the fact that $P_V^N N_2(\gamma, u^N, B^N) = N_2(\gamma, u^N, B^N) = 0$ on the boundary, and (5.9).

Due to the Lemma 7, $N_1(\Pi, f, g) \in U_2(\mathcal{C})$ and $N_2(\gamma, f, g) \in V_2(\mathcal{C})$ holds, by a similar argument above which implies that

$$\begin{aligned} -\Delta(P_U^N N_1(\Pi, u^N, B^N)) &= P_U^N(-\Delta N_1(\Pi, u^N, B^N)) \\ -\Delta(P_V^N N_2(\gamma, u^N, B^N)) &= P_V^N(-\Delta N_2(\gamma, u^N, B^N)) \end{aligned} \quad (5.12)$$

By operating curl^2 to (4.1) and (4.12), taking inner product with $\text{curl}^2 u^N$ and $\text{curl}^2 B^N$, respectively, and integrating it on \mathcal{C} , owing to (5.12), we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\text{curl}^2 u^N|^2 dx + \mu \int_{\mathcal{C}} |\text{curl}^3 u^N|^2 dx \\ &= \int_{\mathcal{C}} \text{curl}^2 P_U^N N_1(\Pi, u^N, B^N) \cdot \text{curl}^2 u^N dx \\ &= \int_{\mathcal{C}} \text{curl}^2 N_1(\Pi, u^N, B^N) \cdot \text{curl}^2 u^N dx \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\text{curl}^2 B^N|^2 dx + \eta \int_{\mathcal{C}} |\text{curl}^3 B^N|^2 dx \\ &= \int_{\mathcal{C}} \text{curl}^2 P_V^N N_2(\gamma, u^N, B^N) \cdot \text{curl}^2 B^N dx \\ &= \int_{\mathcal{C}} \text{curl}^2 N_2(\gamma, u^N, B^N) \cdot \text{curl}^2 B^N dx. \end{aligned} \quad (5.14)$$

By operating curl^2 to (4.1) and (4.12), taking inner product with $\text{curl}^4 u^N$ and $\text{curl}^4 B^N$, respectively, and integrating it on \mathcal{C} , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\text{curl}^3 u^N|^2 dx + \mu \int_{\mathcal{C}} |\text{curl}^4 u^N|^2 dx \\ &= \int_{\mathcal{C}} \text{curl}^2 N_1(\Pi, u^N, B^N) \cdot \text{curl}^4 u^N dx \end{aligned} \quad (5.15)$$

and

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\text{curl}^3 B^N|^2 dx + \eta \int_{\mathcal{C}} |\text{curl}^4 B^N|^2 dx \\ &= \int_{\mathcal{C}} \text{curl}^2 N_2(\gamma, u^N, B^N) \cdot \text{curl}^4 B^N dx. \end{aligned} \quad (5.16)$$

Adding the above equations (5.10)–(5.11) and (5.13)–(5.16) together then gives

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\text{curl}^\alpha u^N|^2 dx + \mu \int_{\mathcal{C}} |\text{curl}^{\alpha+1} u^N|^2 dx \\ &+ \frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\text{curl}^\alpha B^N|^2 dx + \eta \int_{\mathcal{C}} |\text{curl}^{\alpha+1} B^N|^2 dx \\ &= \sum_i^5 J_i^\alpha, \end{aligned} \quad (5.17)$$

where for $\alpha = 1, 2$,

$$\begin{aligned} J_1^\alpha &= - \int_{\mathcal{C}} \text{curl}^\alpha (\text{curl} u^N \times u^N) \cdot (\text{curl}^\alpha u^N) dx, \\ J_2^\alpha &= \int_{\mathcal{C}} \text{curl}^\alpha (\text{curl} B^N \times B^N) \cdot (\text{curl}^\alpha u^N) dx, \\ J_3^\alpha &= \int_{\mathcal{C}} \text{curl}^{\alpha+1} (u^N \times B^N) \cdot (\text{curl}^\alpha B^N) dx, \end{aligned}$$

$$J_4^\alpha = - \int_{\mathcal{C}} \operatorname{curl}^{\alpha+1} (\operatorname{curl} B^N \times B^N) \cdot (\operatorname{curl}^\alpha B^N) dx,$$

$$J_5^\alpha = -\gamma \int_{\mathcal{C}} \operatorname{curl}^{\alpha+1} (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot (\operatorname{curl}^\alpha B^N) dx,$$

and for $\alpha = 3$,

$$J_1^\alpha = - \int_{\mathcal{C}} \operatorname{curl}^{\alpha-1} (\operatorname{curl} u^N \times u^N) \cdot (\operatorname{curl}^{\alpha+1} u^N) dx,$$

$$J_2^\alpha = \int_{\mathcal{C}} \operatorname{curl}^{\alpha-1} (\operatorname{curl} B^N \times B^N) \cdot (\operatorname{curl}^{\alpha+1} u^N) dx,$$

$$J_3^\alpha = \int_{\mathcal{C}} \operatorname{curl}^\alpha (u^N \times B^N) \cdot (\operatorname{curl}^{\alpha+1} B^N) dx,$$

$$J_4^\alpha = - \int_{\mathcal{C}} \operatorname{curl}^\alpha (\operatorname{curl} B^N \times B^N) \cdot (\operatorname{curl}^{\alpha+1} B^N) dx,$$

$$J_5^\alpha = -\gamma \int_{\mathcal{C}} \operatorname{curl}^\alpha (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot (\operatorname{curl}^{\alpha+1} B^N) dx.$$

The higher-order energy estimates will play a crucial role for our purpose in this section. We now go to the estimate for $J_1^\alpha - J_5^\alpha$ terms in the case of $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$ (except for J_5^3).

5.1.1. Higher-Order Energy Estimates for $\alpha = 1$. First, we consider the case of $\alpha = 1$. To estimate J_1^1 , we first note that Lemma 1, and we have

$$\begin{aligned} |J_1^1| &= \left| \int_{\mathcal{C}} \operatorname{curl} (\operatorname{curl} u^N \times u^N) \cdot (\operatorname{curl} u^N) dx \right| \\ &= \left| \int_{\mathcal{C}} (\operatorname{curl} u^N \times u^N) \cdot \operatorname{curl}^2 u^N dx \right| \\ &\lesssim \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|u^N(t, \cdot)\|_{H^1(\mathcal{C})} \|u^N(t, \cdot)\|_{L^\infty(\mathcal{C})} \\ &\lesssim \|u^N(t, \cdot)\|_{H^1(\mathcal{C})}^{3/2} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{3/2}, \end{aligned} \quad (5.18)$$

uniformly in N . For J_2^1 , we get

$$\begin{aligned} |J_2^1| &= \left| \int_{\mathcal{C}} \operatorname{curl} (\operatorname{curl} B^N \times B^N) \cdot (\operatorname{curl} u^N) dx \right| \\ &= \left| \int_{\mathcal{C}} (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^2 u^N dx \right| \\ &\lesssim \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \|B^N(t, \cdot)\|_{L^\infty(\mathcal{C})} \\ &\lesssim \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{3/2} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2}, \end{aligned} \quad (5.19)$$

the term $J_3^1 - J_4^1$ can be estimated as follows.

$$\begin{aligned} |J_3^1| &= \left| \int_{\mathcal{C}} \operatorname{curl}^2 (u^N \times B^N) \cdot \operatorname{curl} B^N dx \right| \\ &\lesssim \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{L^\infty(\mathcal{C})} \\ &\quad + \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \|Du^N(t, \cdot)\|_{L^4(\mathcal{C})} \|DB^N(t, \cdot)\|_{L^4(\mathcal{C})} \\ &\lesssim \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{3/2} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2} \\ &\quad + \|u^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/4} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{3/4} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{5/4} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{3/4} \end{aligned} \quad (5.20)$$

$$\begin{aligned}
|J_4^1| &= \left| \int_{\mathcal{C}} \operatorname{curl}^2 (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl} B^N dx \right| \\
&= \left| \int_{\mathcal{C}} (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^3 B^N dx \right| \\
&\lesssim \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \|B^N(t, \cdot)\|_{L^\infty(\mathcal{C})} \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \\
&\lesssim \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \frac{C}{\epsilon} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^3 \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}
\end{aligned} \tag{5.21}$$

uniformly in N . Finally for J_5^1 ,

$$\begin{aligned}
|J_5^1| &= \gamma \left| \int_{\mathcal{C}} \operatorname{curl}^2 (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl} B^N dx \right| \\
&= \gamma \left| \int_{\mathcal{C}} (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl}^3 B^N dx \right| \\
&\lesssim \gamma \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|B^N(t, \cdot)\|_{L^\infty(\mathcal{C})}^2 \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \\
&\lesssim \gamma \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \gamma \frac{C}{\epsilon} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^4 \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^2
\end{aligned} \tag{5.22}$$

uniformly in N . In (5.21) and (5.22), we use the facts (5.7) and (5.8), respectively. Combining these estimates (5.18)–(5.22), we deduce the following lemma.

Lemma 8. *For $(u^N(t, \cdot), B^N(t, \cdot)) \in W_0^N(\mathcal{C}) \cap W_2(\mathcal{C})$, the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\operatorname{curl} u^N|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\operatorname{curl} B^N|^2 dx + \mu \int_{\mathcal{C}} |\operatorname{curl}^2 u^N|^2 dx + \eta \int_{\mathcal{C}} |\operatorname{curl}^2 B^N|^2 dx \\
&\lesssim \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + Q_1 \left(\|u^N(t, \cdot)\|_{H^2(\mathcal{C})}, \|B^N(t, \cdot)\|_{H^2(\mathcal{C})} \right)
\end{aligned}$$

uniformly in N , where Q_1 is some polynomial in which the degree of each monomial ≥ 2 .

5.1.2. Higher-Order Energy Estimates for $\alpha = 2$. We now turn to the case of $\alpha = 2$. We assume $(u^N, B^N) \in W_0^N(\mathcal{C}) \cap W_2(\mathcal{C})$. We can proceed for $J_1^2 - J_5^2$ by applying Lemma 1–Lemma 3 and Lemma 7. For $J_1^2 - J_3^2$, we obtain

$$\begin{aligned}
|J_1^2| &= \left| \int_{\mathcal{C}} \operatorname{curl}^2 (\operatorname{curl} u^N \times u^N) \cdot \operatorname{curl}^2 u^N dx \right| \\
&\lesssim \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|u^N(t, \cdot)\|_{H^3(\mathcal{C})} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2} \|u^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \\
&\quad + \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|u^N(t, \cdot)\|_{W^{1,4}(\mathcal{C})} \|u^N(t, \cdot)\|_{W^{2,4}(\mathcal{C})} \\
&\lesssim \epsilon \|u^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \frac{C}{\epsilon} \left(\|u^N(t, \cdot)\|_{H^1(\mathcal{C})} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^3 + \|u^N(t, \cdot)\|_{H^1(\mathcal{C})}^{2/5} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{16/5} \right)
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
|J_2^2| &= \left| \int_{\mathcal{C}} \operatorname{curl}^2 (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^2 u^N dx \right| \\
&\lesssim \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \\
&\quad + \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{1,4}(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{2,4}(\mathcal{C})} \\
&\lesssim \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \frac{C}{\epsilon} \left(\|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^2 \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})} \right. \\
&\quad \left. + \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{8/5} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{2/5} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{8/5} \right)
\end{aligned} \tag{5.24}$$

and,

$$\begin{aligned}
|J_3^2| &= \left| \int_{\mathcal{C}} \operatorname{curl}^3 (u^N \times B^N) \cdot \operatorname{curl}^2 B^N dx \right| \\
&= \left| \int_{\mathcal{C}} \operatorname{curl}^2 (u^N \times B^N) \cdot \operatorname{curl}^3 B^N dx \right| \\
&\lesssim \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2} \|u^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \\
&\quad + \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \\
&\quad + \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|u^N(t, \cdot)\|_{W^{1,4}(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{1,4}(\mathcal{C})} \\
&\lesssim \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \frac{C}{\epsilon} \left(\|u^N(t, \cdot)\|_{H^1(\mathcal{C})} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^2 \right. \\
&\quad + \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^2 \|B^N(t, \cdot)\|_{H^1(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})} \\
&\quad \left. + \|u^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \|u^N(t, \cdot)\|_{H^2(\mathcal{C})}^{3/2} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{3/2} \right) \quad (5.25)
\end{aligned}$$

uniformly in N . We also provide the estimate for the term J_4^2 . Applying Lemma 7, and using the following identity

$$\operatorname{curl}^2 (\operatorname{curl} B^N \times B^N) = -\Delta (\operatorname{curl} B^N \times B^N) + \nabla (\operatorname{div} (\operatorname{curl} B^N \times B^N)),$$

we obtain

$$\begin{aligned}
\int_{\mathcal{C}} \operatorname{curl}^3 (\operatorname{curl} B^N \times B^N) \cdot (\operatorname{curl}^2 B^N) dx &= \int_{\mathcal{C}} \operatorname{curl}^2 (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^3 B^N dx \\
&= - \int_{\mathcal{C}} \Delta (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^3 B^N dx + \int_{\mathcal{C}} \nabla (\operatorname{div} (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl}^3 B^N dx.
\end{aligned}$$

Integrating the second terms on the right-hand side yields

$$\begin{aligned}
&\int_{\mathcal{C}} \nabla (\operatorname{div} (\operatorname{curl} B^N \times B^N)) \cdot (\operatorname{curl}^3 B^N) dx \\
&= - \int_{\mathcal{C}} (\operatorname{div} (\operatorname{curl} B^N \times B^N)) \cdot (\operatorname{div} \operatorname{curl}^3 B^N) dx + \int_{\partial \mathcal{C}} \operatorname{div} (\operatorname{curl} B^N \times B^N) (\operatorname{curl}^3 B^N \cdot \vec{n}) ds \\
&= \int_{\partial \mathcal{C}} \operatorname{div} (\operatorname{curl} B^N \times B^N) (\operatorname{curl}^3 B^N \cdot \vec{n}) ds.
\end{aligned}$$

For $x = (x_1, x_2, x_3)$ and any basis element $\beta_k(x) = ((\beta_k)_{x_1}, (\beta_k)_{x_2}, (\beta_k)_{x_3})$ in $W_0(\mathcal{C})$,

$$\operatorname{curl}^2 \beta_k(x) = \lambda_{2,k} \beta_k(x) \text{ for some } \lambda_{2,k} \in \mathbb{R}, \quad (5.26)$$

$$\operatorname{curl} \beta_k(x) \cdot \vec{n} = - \left(\frac{\partial (\beta_k)_{x_2}}{\partial x_3}, -\frac{\partial (\beta_k)_{x_1}}{\partial x_3}, 0 \right) \cdot (0, 0, 1) = 0 \text{ on } \partial \mathcal{C}. \quad (5.27)$$

Using (5.26)–(5.27), we can compute

$$\operatorname{curl}^3 B^N \cdot \vec{n} = 0 \text{ on } \partial \mathcal{C}, \quad (5.28)$$

since B^N is a linear combination of $\beta_k(x)$. Using (5.28), we have

$$\int_{\mathcal{C}} \operatorname{curl}^2 (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^3 B^N dx = - \int_{\mathcal{C}} \Delta (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl} \Delta B^N dx,$$

which gives us to write J_4^2 as follows:

$$\begin{aligned}
J_4^2 &= \int_{\mathcal{C}} \operatorname{curl}^2 (\operatorname{curl} B^N \times B^N) \cdot \operatorname{curl}^3 B^N dx \\
&= - \int_{\mathcal{C}} \Delta (\operatorname{curl} B^N \times B^N) \cdot \Delta \operatorname{curl} B^N dx \\
&= - \int_{\mathcal{C}} (\Delta \operatorname{curl} B^N \times B^N) \cdot \Delta \operatorname{curl} B^N dx \\
&\quad - \int_{\mathcal{C}} (\Delta (\operatorname{curl} B^N \times B^N) - (\Delta \operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl} \Delta B^N dx \\
&= - \int_{\mathcal{C}} (\Delta (\operatorname{curl} B^N \times B^N) - (\Delta \operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl} \Delta B^N dx.
\end{aligned}$$

Thus we can obtain

$$\begin{aligned}
|J_4^2| &= \left| \int_{\mathcal{C}} (\Delta (\operatorname{curl} B^N \times B^N) - (\operatorname{curl} \Delta B^N \times B^N)) \cdot \operatorname{curl} \Delta B^N dx \right| \\
&\lesssim \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{2,4}(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{1,4}(\mathcal{C})} \\
&\lesssim \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \frac{C}{\epsilon} \left(\|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^2 \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^8 \right)
\end{aligned} \tag{5.29}$$

uniformly in N . In a similar way, we can estimate J_5^2 as follows:

$$\begin{aligned}
J_5^2 &= -\gamma \int_{\mathcal{C}} \operatorname{curl}^3 (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl}^2 B^N dx \\
&= -\gamma \int_{\mathcal{C}} \operatorname{curl}^2 (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl}^3 B^N dx \\
&= \gamma \int_{\mathcal{C}} \Delta (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl}^3 B^N dx \\
&\quad - \gamma \int_{\mathcal{C}} \nabla (\operatorname{div} (B^N \times (\operatorname{curl} B^N \times B^N))) \cdot \operatorname{curl}^3 B^N dx.
\end{aligned}$$

In view of (5.28), we have

$$\begin{aligned}
&\int_{\mathcal{C}} \nabla (\operatorname{div} (B^N \times (\operatorname{curl} B^N \times B^N))) \cdot \operatorname{curl}^3 B^N dx \\
&= \int_{\mathcal{C}} (\operatorname{div} (B^N \times (\operatorname{curl} B^N \times B^N))) (\operatorname{div} \operatorname{curl}^3 B^N) dx \\
&\quad + \int_{\partial \mathcal{C}} (\operatorname{div} (B^N \times (\operatorname{curl} B^N \times B^N))) (\operatorname{curl}^3 B^N \cdot \vec{n}) ds \\
&= \int_{\partial \mathcal{C}} (\operatorname{div} (B^N \times (\operatorname{curl} B^N \times B^N))) (\operatorname{curl}^3 B^N \cdot \vec{n}) ds \\
&= 0.
\end{aligned}$$

We then derive

$$\begin{aligned}
J_5^2 &= \gamma \int_{\mathcal{C}} \Delta (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \operatorname{curl}^3 B^N dx \\
&= -\gamma \int_{\mathcal{C}} (B^N \times (\Delta \operatorname{curl} B^N \times B^N)) \cdot \Delta \operatorname{curl} B^N dx + J_5^{2,1},
\end{aligned}$$

where

$$J_5^{2,1} = -\gamma \int_{\mathcal{C}} \Delta (B^N \times (\operatorname{curl} B^N \times B^N)) \cdot \Delta \operatorname{curl} B^N dx$$

$$+ \gamma \int_{\mathcal{C}} (B^N \times (\Delta \operatorname{curl} B^N \times B^N)) \cdot \Delta \operatorname{curl} B^N dx.$$

We can show that,

$$\begin{aligned} & |J_5^{2,1}| \\ & \lesssim \gamma \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{1/2} \|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^{1/2} \|B^N(t, \cdot)\|_{W^{2,4}(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{1,4}(\mathcal{C})} \\ & \quad + \gamma \|B^N(t, \cdot)\|_{H^3(\mathcal{C})} \|B^N(t, \cdot)\|_{W^{1,6}(\mathcal{C})}^3. \end{aligned}$$

Thus,

$$\begin{aligned} & J_5^2 + \gamma \int_{\mathcal{C}} |\Delta \operatorname{curl} B^N \times B^N|^2 dx \\ & \lesssim \epsilon \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \frac{C}{\epsilon} \left(\|B^N(t, \cdot)\|_{H^1(\mathcal{C})}^6 \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^{12} + \|B^N(t, \cdot)\|_{H^2(\mathcal{C})}^6 \right). \end{aligned} \quad (5.30)$$

Combining (5.23)–(5.25) and (5.29)–(5.30), we can prove the following lemma.

Lemma 9. For $(u^N, B^N) \in W_0^N(\mathcal{C}) \cap W_2(\mathcal{C})$, it holds

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\operatorname{curl}^2 u^N|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathcal{C}} |\operatorname{curl}^2 B^N|^2 dx \\ & \quad + \mu \int_{\mathcal{C}} |\operatorname{curl}^3 u^N|^2 dx + \eta \int_{\mathcal{C}} |\operatorname{curl}^3 B^N|^2 dx + \gamma \int_{\mathcal{C}} |\Delta \operatorname{curl} B^N \times B^N|^2 dx \\ & \lesssim \epsilon \left(\|u^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 + \|B^N(t, \cdot)\|_{H^3(\mathcal{C})}^2 \right) + Q_2 \left(\|u^N(t, \cdot)\|_{H^2(\mathcal{C})}, \|B^N(t, \cdot)\|_{H^2(\mathcal{C})} \right) \end{aligned}$$

uniformly in N , where Q_2 is some polynomial in which degree of each monomial ≥ 2 .

The above lemma will be used to prove the local well-posedness for large initial data in $H^2(\mathcal{C})$ and global well-posedness for small initial data in $H^2(\mathcal{C})$ respectively.

5.1.3. Higher-Order Energy Estimates for $\alpha = 3$. We ignore the ion-slip effect ($\gamma = 0$). By a similar manner to Lemma 8 and Lemma 9, we can also obtain estimations for $\alpha = 3$ which will be useful for the vanishing viscosity limit $\mu \rightarrow 0$. In fact, we obtain the following estimate for J_1^3 – J_3^3

$$\begin{aligned} & \sum_{k=1}^3 |J_k^3| \lesssim \frac{C}{\epsilon} \left(\|\operatorname{curl}^3 u^N\|_{L^2(\mathcal{C})}^2 + \|\operatorname{curl}^3 B^N\|_{L^2(\mathcal{C})}^2 \right)^2 \\ & \quad + \epsilon \left(\|\operatorname{curl}^3 u^N\|_{L^2(\mathcal{C})}^2 + \|\operatorname{curl}^3 B^N\|_{L^2(\mathcal{C})}^2 \right), \end{aligned} \quad (5.31)$$

which is derived by similar arguments in the proof of Proposition 5.1 in [40], and thus further details are skipped. With the help of Lemma 7, we estimate J_4^3 which deal with the Hall effect. By using vector calculus identities (2.1), we obtain

$$\begin{aligned} \operatorname{curl}^3 (\operatorname{curl} B^N \times B^N) &= \operatorname{curl}^2 ((B^N \cdot \nabla) \operatorname{curl} B^N - (\operatorname{curl} B^N \cdot \nabla) B^N) \\ &= (B^N \cdot \nabla) \operatorname{curl}^3 B^N + R_{1,1}, \\ (B^N \cdot \nabla) \operatorname{curl}^3 B^N &= -B^N \times \operatorname{curl}^4 B^N + \nabla (\operatorname{curl}^3 B^N \cdot B^N) \\ &\quad + R_{1,2}, \end{aligned}$$

where $R_{1,1}$ and $R_{1,2}$ consist of nonlinear terms for the spatial derivatives $D^k B^N$ for $0 \leq k \leq 3$. It is obvious that

$$\int_{\mathcal{C}} (\operatorname{curl}^4 B^N \times B^N) \cdot (\operatorname{curl}^4 B^N) dx = 0,$$

and

$$\int_C \nabla (\operatorname{curl}^3 B^N \cdot B^N) \cdot (\operatorname{curl}^4 B^N) dx = 0 \quad (\text{by integration by parts}).$$

Hence,

$$|J_4^3| \leq \left| \int_C R_{1,1} \cdot \operatorname{curl}^4 B^N dx \right| + \left| \int_C R_{1,2} \cdot \operatorname{curl}^4 B^N dx \right|.$$

We can estimate

$$\left| \int_C R_{1,1} \cdot \operatorname{curl}^4 B^N dx \right| + \left| \int_C R_{1,2} \cdot \operatorname{curl}^4 B^N dx \right| \lesssim \|\operatorname{curl}^4 B\|_{L^2(C)} \|\operatorname{curl}^3 B\|_{L^2(C)}^2,$$

by a similar fashion to the estimations in Sect. 5.1.2. Hence, we can deduce

$$|J_4^3| \lesssim \|\operatorname{curl}^4 B\|_{L^2(C)} \|\operatorname{curl}^3 B\|_{L^2(C)}^2. \quad (5.32)$$

Combining together (5.31) and (5.32), the following lemma is obtained.

Lemma 10. For $(u^N, B^N) \in W_0^N(C) \cap W_3(C)$, it holds

$$\begin{aligned} \sum_{k=1}^4 |J_k^3| &\lesssim \frac{C}{\epsilon} \left(\|\operatorname{curl}^3 u^N\|_{L^2(C)}^2 + \|\operatorname{curl}^3 B^N\|_{L^2(C)}^2 + \|\operatorname{curl}^4 B^N\|_{L^2(C)}^2 \right)^2 \\ &\quad + \epsilon \left(\|\operatorname{curl}^3 u^N\|_{L^2(C)}^2 + \|\operatorname{curl}^3 B^N\|_{L^2(C)}^2 + \|\operatorname{curl}^4 B^N\|_{L^2(C)}^2 \right). \end{aligned}$$

5.2. The Proofs from Theorem 2 (A) to Theorem 4 (A)

Using the estimates in Lemma 8–9, we now prove Theorem 2 (A).

Proof of Theorem 2 (A). Let $(u^0, B^0) \in W_2(C)$. By using Lemma 8 and Lemma 9 to (5.17), we can derive the following inequality with the help of Young's interpolation inequality:

$$\begin{aligned} &\frac{1}{2} \sum_{k=0}^2 \frac{d}{dt} \left(\left\| \operatorname{curl}^k u^N(t, \cdot) \right\|_{L^2(C)}^2 + \left\| \operatorname{curl}^k B^N(t, \cdot) \right\|_{L^2(C)}^2 \right) \\ &\quad + \sum_{k=1}^3 \left(\mu \left\| \operatorname{curl}^k u^N(t, \cdot) \right\|_{L^2(C)}^2 + \eta \left\| \operatorname{curl}^k B^N(t, \cdot) \right\|_{L^2(C)}^2 \right) \\ &\quad + \gamma \left(\left\| \operatorname{curl} B^N(t, \cdot) \times B^N(t, \cdot) \right\|_{L^2(C)}^2 + \left\| \operatorname{curl}^3 B^N(t, \cdot) \times B^N(t, \cdot) \right\|_{L^2(C)}^2 \right) \\ &\leq Q \left(\|u^N(t, \cdot)\|_{H^2(C)}, \|B^N(t, \cdot)\|_{H^2(C)} \right), \end{aligned} \quad (5.33)$$

for some nonnegative continuous function $Q(x_1, x_2)$ which is independent of N in $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$. Hence, there exists $T_0 > 0$ such that (u^N, B^N) is bounded in $L^\infty([0, T_0]; W_2(C)) \cap L^2([0, T_0]; W_3(C))$:

$$\begin{aligned} \|u^N(t, \cdot)\|_{H^2(C)}^2 + \|B^N(t, \cdot)\|_{H^2(C)}^2 &\leq 2 \left(\|u^N(0, \cdot)\|_{H^2(C)}^2 + \|B^N(0, \cdot)\|_{H^2(C)}^2 \right) \\ &\leq 2 \left(\|u^0(\cdot)\|_{H^2(C)}^2 + \|B^0(\cdot)\|_{H^2(C)}^2 \right), \end{aligned} \quad (5.34)$$

for all $t \in [0, T_0]$, and

$$\int_0^{T_0} \|u^N(t, \cdot)\|_{H^3(C)}^2 + \|B^N(t, \cdot)\|_{H^3(C)}^2 dx \leq 4 \left(\|u^N(0, \cdot)\|_{H^2(C)}^2 + \|B^N(0, \cdot)\|_{H^2(C)}^2 \right) \quad (5.35)$$

where (u^0, B^0) is an initial data. Moreover, we can also show $\left(\frac{\partial u^N}{\partial t}, \frac{\partial B^N}{\partial t} \right)$ is bounded in $L^2([0, T_0]; W_1(C))$. Then we can find a strong solution by taking limit $N \rightarrow \infty$. Next we prove the uniqueness for initial data in $W_2(C)$. Define $u_d = u_2 - u_1$ and $B_d = B_2 - B_1$, where (u_1, B_1) and (u_2, B_2) are solutions with the same initial data $(u^0, B^0) \in W_2(C)$. Then we deduce

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|u_d(t, \cdot)\|_{L^2(C)}^2 + \|B_d(t, \cdot)\|_{L^2(C)}^2 \right) + \mu \|\operatorname{curl} u_d(t, \cdot)\|_{L^2(C)}^2 + \eta \|\operatorname{curl} B_d(t, \cdot)\|_{L^2(C)}^2 \\
& = K_1 + K_2 + K_3 + K_4,
\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}
K_1 &= - \int_C u_d \cdot (\operatorname{curl} u_d \times u_2 + \operatorname{curl} u_1 \times u_d - \operatorname{curl} B_d \times B_2 - \operatorname{curl} B_1 \times B_d) dx, \\
K_2 &= - \int_C B_d \cdot (\operatorname{curl} (B_d \times u_2) + \operatorname{curl} (B_1 \times u_d)) dx, \\
K_3 &= - \int_C \operatorname{curl} B_d \cdot (\operatorname{curl} B_d \times B_2 + \operatorname{curl} B_1 \times B_d) dx, \\
K_4 &= - \gamma \int_C (B_2 \times (\operatorname{curl} B_d \times B_2 + \operatorname{curl} B_1 \times B_2) - B_1 \times (\operatorname{curl} B_1 \times B_1)) \cdot \operatorname{curl} B_d dx \\
&= - \gamma \int_C |\operatorname{curl} B_d \times B_2|^2 dx \\
&\quad - \gamma \int_C (B_d \times \operatorname{curl} B_1 \times B_2 + B_1 \times \operatorname{curl} B_1 \times B_d) \cdot \operatorname{curl} B_d dx.
\end{aligned}$$

Thanks to Young's interpolation inequality, we have

$$\begin{aligned}
|K_1| &\lesssim \|u_d\|_{L^2(C)} \|\operatorname{curl} u_d\|_{L^2(C)} \|u_2\|_{L^\infty(C)} + \|u_d\|_{L^2(C)} \|\operatorname{curl} B_d\|_{L^2(C)} \|B_2\|_{L^\infty(C)} \\
&\quad + \|u_d\|_{L^2(C)} \|B_d\|_{L^2(C)} \|\operatorname{curl} B_1\|_{L^\infty(C)} \\
&\lesssim \epsilon \left(\|\operatorname{curl} u_d\|_{L^2(C)}^2 + \|\operatorname{curl} B_d\|_{L^2(C)}^2 \right) \\
&\quad + \frac{C}{\epsilon} \left(\|u_2\|_{H^2(C)}^2 + \|B_2\|_{H^2(C)}^2 + \|B_1\|_{H^3(C)}^2 + 1 \right) \left(\|u_d\|_{L^2(C)}^2 + \|B_d\|_{L^2(C)}^2 \right)
\end{aligned} \tag{5.37}$$

and

$$\begin{aligned}
|K_2| &\lesssim \|B_d\|_{L^2(C)} \|\operatorname{curl} B_d\|_{L^2(C)} \|u_2\|_{L^\infty(C)} + \|u_d\|_{L^2(C)} \|B_d\|_{L^2(C)} \|\operatorname{curl} B_1\|_{L^\infty(C)} \\
&\quad + \|B_d\|_{L^2(C)} \|\operatorname{curl} u_d\|_{L^2(C)} \|B_1\|_{L^\infty(C)} \\
&\lesssim \epsilon \left(\|\operatorname{curl} u_d\|_{L^2(C)}^2 + \|\operatorname{curl} B_d\|_{L^2(C)}^2 \right) \\
&\quad + \frac{C}{\epsilon} \left(\|u_2\|_{H^2(C)}^2 + \|B_1\|_{H^2(C)}^2 + \|B_1\|_{H^3(C)}^2 + 1 \right) \left(\|u_d\|_{L^2(C)}^2 + \|B_d\|_{L^2(C)}^2 \right).
\end{aligned} \tag{5.38}$$

For K_3 and K_4 , we obtain

$$\begin{aligned}
|K_3| &\lesssim \|B_d\|_{L^2(C)} \|\operatorname{curl} B_d\|_{L^2(C)} \|\operatorname{curl} B_1\|_{L^\infty(C)} \\
&\lesssim \epsilon \|\operatorname{curl} B_d\|_{L^2(C)}^2 + \frac{C}{\epsilon} \|B_1\|_{H^3(C)}^2 \|B_d\|_{L^2(C)}^2
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
|K_4| &\lesssim \|B_2\|_{L^\infty(C)} \|\operatorname{curl} B_1\|_{L^4(C)} \|B_d\|_{L^4(C)} \|\operatorname{curl} B_d\|_{L^2(C)} \\
&\quad + \|B_1\|_{L^\infty(C)} \|\operatorname{curl} B_1\|_{L^4(C)} \|B_d\|_{L^4(C)} \|\operatorname{curl} B_d\|_{L^2(C)} \\
&\lesssim \epsilon \|\operatorname{curl} B_d\|_{L^2(C)}^2 + \frac{C}{\epsilon} \left(\|B_1\|_{H^2(C)}^{16} + \|B_2\|_{H^2(C)}^{16} \right) \|B_d\|_{L^2(C)}^2.
\end{aligned} \tag{5.40}$$

Therefore, collecting (5.37)–(5.40) and applying (5.34)–(5.35), we conclude that

$$\begin{aligned}
& \frac{d}{dt} \left(\|u_d(t, \cdot)\|_{L^2(C)}^2 + \|B_d(t, \cdot)\|_{L^2(C)}^2 \right) \\
& \quad + \left(\mu \|\operatorname{curl} u_d(t, \cdot)\|_{L^2(C)}^2 + \eta \|\operatorname{curl} B_d(t, \cdot)\|_{L^2(C)}^2 \right) \\
& \quad + \gamma \|B_2(t, \cdot) \times \operatorname{curl} B_d(t, \cdot)\|_{L^2(C)}^2
\end{aligned}$$

$$\leq F \left(\|u_d(t, \cdot)\|_{L^2(C)}^2 + \|B_d(t, \cdot)\|_{L^2(C)}^2 \right), \quad (5.41)$$

where

$$F = F \left(\|u^{1,2}(\cdot)\|_{H^2(C)}^2, \|B^{1,2}(\cdot)\|_{H^2(C)}^2, \|u^{1,2}(\cdot)\|_{H^3(C)}^2, \|B^{1,2}(\cdot)\|_{H^3(C)}^2 \right) > 0, \quad (5.42)$$

is an integrable function of time. Hence, using Grownwall's inequality, we get the uniqueness of solutions for the given initial data in $W_2(C)$. This completes the proof of Theorem 2 (A). \square

Moreover, applying Lemma 8 and 9, we show the global well-posedness for small initial data.

Proof of Theorem 3 (A). By applying Lemma 8 and Lemma 9 to (5.17), and the local-in-time well-posedness stated in Theorem 2 (A), we also easily get

$$\begin{aligned} & \frac{d}{dt} \sum_{k=0}^2 \left(\left\| \operatorname{curl}^k u(t, \cdot) \right\|_{L^2(C)}^2 + \left\| \operatorname{curl} B(t, \cdot) \right\|_{L^2(C)}^2 \right) + \sum_{k=1}^3 \left(\mu \left\| \operatorname{curl}^k u(t, \cdot) \right\|_{L^2(C)}^2 + \eta \left\| \operatorname{curl}^k B(t, \cdot) \right\|_{L^2(C)}^2 \right) \\ & + \gamma \left(\|B \times \operatorname{curl} B(t, \cdot)\|_{L^2(C)}^2 + \|B \times \operatorname{curl}^3 B(t, \cdot)\|_{L^2(C)}^2 \right) \\ & \leq Q \left(\|u(t, \cdot)\|_{H^2(C)}^2 + \|B(t, \cdot)\|_{H^2(C)}^2 \right), \end{aligned} \quad (5.43)$$

where Q_2 is some polynomial in which degree of each monomial ≥ 2 . Noting that the Poincare-like inequalities,

$$\|u(t, \cdot)\|_{H^2(C)} \lesssim \|\operatorname{curl} u(t, \cdot)\|_{H^2(C)}, \quad \|B(t, \cdot)\|_{H^2(C)} \lesssim \|\operatorname{curl} B(t, \cdot)\|_{H^2(C)},$$

we see that if initial data is sufficiently small,

$$\frac{d}{dt} \left(\|u(t, \cdot)\|_{H^2(C)}^2 + \|B(t, \cdot)\|_{H^2(C)}^2 \right) \leq 0,$$

which establishes the global well-posedness result. This completes the proof of Theorem 3 (A). \square

Applying Lemma 10, we can show the vanishing viscosity limit $\mu \rightarrow 0$ for $\gamma = 0$.

Proof of Theorem 4 (A). Let $(u_{\mu, \eta}, B_{\mu, \eta}) \in L^\infty([0, T]; W_2(C)) \cap L^2([0, T]; W_3(C))$ be a solution of (1.1)–(1.4) for $\mu, \eta > 0$ with (1.7)–(1.8). By using Lemma 10 to (5.17), we obtain $T_0 > 0$ such that

$$\begin{aligned} & \left(\|u_{\mu, \eta}(t, \cdot)\|_{H^3(C)}^2 + \|B_{\mu, \eta}(t, \cdot)\|_{H^3(C)}^2 \right) + \int_0^t \|B(\tau, \cdot)\|_{H^4(C)}^2 d\tau \leq C, \\ & \int_0^t \left(\left\| \frac{\partial u_{\mu, \eta}}{\partial t}(\tau, \cdot) \right\|_{H^2(C)}^2 + \left\| \frac{\partial B_{\mu, \eta}}{\partial t}(\tau, \cdot) \right\|_{H^2(C)}^2 \right) d\tau \leq C, \end{aligned}$$

in $0 \leq t \leq T_0$, where C depends on η and initial data in $H^3(C)$ not μ . Applying this result and a similar manner to Theorem 5.2 in [40], there exists a subsequence $\{\mu_k\}_{k=1}^\infty$ of μ and $(u_{0, \eta}, B_{0, \eta})$ such that

$$\begin{aligned} & (u_{\mu_k, \eta}, B_{\mu_k, \eta}) \rightarrow (u_{0, \eta}, B_{0, \eta}) \text{ in } L^q([0, T_0]; H^3(C)), \\ & (u_{\mu_k, \eta}, B_{\mu_k, \eta}) \rightarrow (u_{0, \eta}, B_{0, \eta}) \text{ in } C([0, T_0]; H^2(C)), \end{aligned}$$

for $1 \leq q < \infty$ as $\mu_k \rightarrow 0$. Passing to the limit, we can show that $u_{0, \eta}, B_{0, \eta}$ is a solution of (1.1)–(1.4) for $\mu = 0, \eta > 0$ with (1.7)–(1.8). Uniqueness is also guaranteed. In order to show the convergence rate, we denote $u^d = u_{\mu, \eta} - u_{0, \eta}$ and $B^d = B_{\mu, \eta} - B_{0, \eta}$. Then we get

$$\begin{aligned} & \frac{d}{dt} \left(\left\| \operatorname{curl}^2 u^d, \operatorname{curl}^2 B^d \right\|_{L^2(C)}^2 + \eta \left\| \operatorname{curl}^3 B^d(t, \cdot) \right\|_{L^2(C)}^2 \right) \\ & \lesssim \sum_{k=1}^5 |Q_k|, \end{aligned}$$

where

$$\begin{aligned}
 Q_1 &:= \int_C \operatorname{curl}^2 (\operatorname{curl} u_{0,\eta} \times u_{0,\eta} - \operatorname{curl} u_{\mu,\eta} \times u_{\mu,\eta}) \cdot \operatorname{curl}^2 u^d dx, \\
 Q_2 &:= \int_C \operatorname{curl}^2 (\operatorname{curl} B_{\mu,\eta} \times B_{\mu,\eta} - \operatorname{curl} B_{0,\eta} \times B_{0,\eta}) \cdot \operatorname{curl}^2 u^d dx, \\
 Q_3 &:= \int_C (\operatorname{curl}^3 (u_{\mu,\eta} \times B_{\mu,\eta}) - \operatorname{curl}^3 (u_{0,\eta} \times B_{0,\eta})) \cdot \operatorname{curl}^2 B^d dx, \\
 Q_4 &:= -\mu \int_C \operatorname{curl}^3 u_{\mu,\eta} (t, \cdot) \cdot \operatorname{curl}^3 u^d (t, \cdot) dx, \\
 Q_5 &:= \int_C (\operatorname{curl}^2 (\operatorname{curl} B_{\mu,\eta} \times B_{\mu,\eta}) - \operatorname{curl}^2 (\operatorname{curl} B_{0,\eta} \times B_{0,\eta})) \cdot \operatorname{curl}^3 B^d dx.
 \end{aligned}$$

Estimations from Q_1 to Q_4 are derived as follows (see the proof of Theorem 5.3 in [40]):

$$\sum_{k=1}^4 |Q_k| \leq C(T_0) \left(\mu + \|(\operatorname{curl}^2 u^d, \operatorname{curl}^2 B^d)(t, \cdot)\|_{L^2(C)}^2 \right), \quad (5.44)$$

for some constant $C(T_0) > 0$. For Q_5 , since

$$\begin{aligned}
 \|\operatorname{curl}^3 B_{\mu,\eta}(t, \cdot)\|_{L^2(C)}^2 &\lesssim \|\operatorname{curl}^3 B^0(\cdot)\|_{L^2(C)}^2, \\
 \|\operatorname{curl}^3 B_{0,\eta}(t, \cdot)\|_{L^2(C)}^2 &\lesssim \|\operatorname{curl}^3 B^0(\cdot)\|_{L^2(C)}^2,
 \end{aligned}$$

using Young's interpolation inequality, we can deduce that for any $\varepsilon > 0$, and $\mu > 0, \eta > 0$, there exists $C > 0$ which depends on initial data such that

$$\begin{aligned}
 |Q_5| &\lesssim \|\operatorname{curl}^3 B^d(t, \cdot)\|_{L^2(C)} \|\operatorname{curl}^2 B^d(t, \cdot)\|_{L^2(C)} \|\operatorname{curl}^3 B^0(t, \cdot)\|_{L^2(C)} \\
 &\leq \varepsilon \|\operatorname{curl}^3 B^d(t, \cdot)\|_{L^2(C)}^2 + \frac{C}{\varepsilon} \|\operatorname{curl}^2 B^d(t, \cdot)\|_{L^2(C)}^2.
 \end{aligned} \quad (5.45)$$

Thanks to (5.44)–(5.45), for sufficiently small $\varepsilon > 0$, we can compute

$$\begin{aligned}
 \frac{d}{dt} \|(\operatorname{curl}^2 u^d, \operatorname{curl}^2 B^d)(t, \cdot)\|_{L^2(C)}^2 + \eta \|\operatorname{curl}^3 B^d(t, \cdot)\|_{L^2(C)}^2 + \|(\operatorname{curl}^3 B^d \times B_{\mu,\eta})(t, \cdot)\|_{L^2(C)}^2 \\
 \lesssim C(T_0) \left(\mu + \|(\operatorname{curl}^2 u^d, \operatorname{curl}^2 B^d)(t, \cdot)\|_{L^2(C)}^2 \right),
 \end{aligned}$$

for some $C(T_0) > 0$. Owing to the Gronwall inequality, we can show the convergence rate. This completes the proof of Theorem 4 (A). \square

6. Main Results for the Whole Space \mathbb{R}^3

In this section, we briefly discuss about the our main results in the whole space \mathbb{R}^3 . Let J_ε be an operator such that

$$\begin{aligned}
 J_\varepsilon f(x) &= f(x) * \xi_\varepsilon, \\
 \xi_\varepsilon &= \varepsilon^{-3} \xi\left(\frac{x}{\varepsilon}\right),
 \end{aligned}$$

where $0 \leq \xi \in C_0^\infty(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \xi dx = 1$. Then we consider the following regularized problem:

$$\frac{\partial u_\varepsilon}{\partial t} - \mu \Delta J_\varepsilon^2 u_\varepsilon = -\nabla \Pi_\varepsilon + J_\varepsilon (\operatorname{curl} B_\varepsilon \times B_\varepsilon) - J_\varepsilon (\operatorname{curl} u_\varepsilon \times u_\varepsilon), \quad (6.1)$$

$$\begin{aligned}
 \frac{\partial B_\varepsilon}{\partial t} - \eta \Delta J_\varepsilon^2 B_\varepsilon &= \operatorname{curl} (J_\varepsilon (\operatorname{curl} B_\varepsilon \times B_\varepsilon)) + J_\varepsilon \operatorname{curl} (u_\varepsilon \times B_\varepsilon) \\
 &\quad - \gamma J_\varepsilon \operatorname{curl} ((J_\varepsilon B_\varepsilon \times (\operatorname{curl} J_\varepsilon B_\varepsilon \times J_\varepsilon B_\varepsilon))),
 \end{aligned} \quad (6.2)$$

$$\operatorname{div} u_\varepsilon = 0, \quad \operatorname{div} B_\varepsilon = 0, \quad (6.3)$$

$$u_\varepsilon(0, x) = J_\varepsilon u^0(x), \quad B_\varepsilon(0, x) = J_\varepsilon B^0(x), \quad (6.4)$$

where $u_\varepsilon = J_\varepsilon u$, $B_\varepsilon = J_\varepsilon B$. By multiplying $u_\varepsilon, B_\varepsilon$ to (6.1)–(6.2), and integrating by parts, we obtain the following lemma.

Lemma 11. *Let $u^0(x), B^0(x) \in L^2(\mathbb{R}^3)$. Then there exists a unique global solution $u_\varepsilon, B_\varepsilon \in L^2([0, \infty); H^1(\Omega)) \cap L^\infty([0, \infty); L^2(\Omega))$ of (6.1)–(6.4) such that*

$$\begin{aligned} & \frac{1}{2} \left(\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + \|B_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \right) \\ & + \int_0^t \left(\mu \|\operatorname{div} J_\varepsilon u_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + \eta \|\operatorname{div} J_\varepsilon B_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \right) d\tau \\ & + \gamma \int_0^t \|\operatorname{curl} J_\varepsilon B_\varepsilon(\tau, \cdot) \times J_\varepsilon B_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}^2 d\tau \\ & \leq \frac{1}{2} \left(\|u_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + \|B_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \right). \end{aligned} \quad (6.5)$$

The procedure to estimate (6.5) is similar to estimates in Theorem 1 in a bounded domain Ω . Hence using (6.5) and following similar arguments in [24], we can show the existence of $u_\varepsilon, B_\varepsilon \in L^2([0, \infty); H^1(\Omega)) \cap L^\infty([0, \infty); L^2(\Omega))$. Then we can construct regularized solutions satisfying (6.5) uniformly in ε , and it provides a global weak solution by applying weak convergence and compactness theorems. This shows the proof of Theorem 1 (B) in the case of \mathbb{R}^3 . We can also derive the higher-order energy estimates in a similar manner to [6], i.e., we can obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} \left(|D^\alpha u|^2 + |D^\alpha B|^2 \right) dx + \sum_{0 \leq |\alpha| \leq m} \left(\int_{\mathbb{R}^3} \mu |D^{\alpha+1} u|^2 dx + \eta |D^{\alpha+1} B|^2 \right) dx \\ & := \sum_{k=1}^5 S_k^m. \end{aligned}$$

where

$$\begin{aligned} S_1^m &= - \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha (\operatorname{curl} u \times u) \cdot (D^\alpha u) dx, \\ S_2^m &= \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha (\operatorname{curl} B \times B) \cdot (D^\alpha u) dx, \\ S_3^m &= \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha \operatorname{curl} (u \times B) \cdot (D^\alpha B) dx, \\ S_4^m &= - \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha (\operatorname{curl} B \times B) \cdot (D^\alpha \operatorname{curl} B) dx, \\ S_5^m &= -\gamma \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha (B \times (\operatorname{curl} B \times B)) \cdot (D^\alpha \operatorname{curl} B) dx. \end{aligned}$$

In the whole space \mathbb{R}^3 , we do not worry about the compatibility issue. Therefore, we can increase regularity for any $3 \leq m \in \mathbb{Z}$. Indeed, it is known that for $3 \leq m \in \mathbb{Z}$,

$$\sum_{k=1}^3 |S_k^m| \lesssim \left(\|u(t, \cdot)\|_{H^m(\mathbb{R}^3)}^{3/2} + \|B(t, \cdot)\|_{H^m(\mathbb{R}^3)}^{3/2} \right)^2. \quad (6.6)$$

We can rewrite S_4^m as follows.

$$\begin{aligned} S_4^m &= - \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} (D^\alpha \operatorname{curl} B \times B) \cdot (D^\alpha \operatorname{curl} B) dx \\ &\quad - \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} (D^\alpha (\operatorname{curl} B \times B) - (D^\alpha \operatorname{curl} B \times B)) \cdot (D^\alpha \operatorname{curl} B) dx \\ &= - \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} (D^\alpha (\operatorname{curl} B \times B) - (D^\alpha \operatorname{curl} B \times B)) \cdot (D^\alpha \operatorname{curl} B) dx. \end{aligned}$$

We then estimate

$$|S_4^m| \lesssim \|DB(t, \cdot)\|_{H^m(\mathbb{R}^3)} \|B(t, \cdot)\|_{H^m(\mathbb{R}^3)}^2, \quad (6.7)$$

for any $3 \leq m \in \mathbb{Z}$.

Moreover, we can estimate S_5^m for any $3 \leq m \in \mathbb{Z}$. We first explain the difference between S_5^3 and J_5^3 in (5.17). In the case of J_5^3 in \mathcal{C} , one can deduce

$$\begin{aligned} &\operatorname{curl}^3 (B^N \times (\operatorname{curl} B^N \times B^N)) \\ &= \Delta \operatorname{curl} \left(|B^N|^2 \operatorname{curl} B^N - (B^N \cdot \operatorname{curl} B^N) B^N \right) \\ &= \Delta \left(|B^N|^2 \operatorname{curl}^2 B^N + \nabla |B^N|^2 \operatorname{curl} B^N - (B^N \cdot \operatorname{curl} B^N) \times \operatorname{curl} B^N - \nabla (B^N \cdot \operatorname{curl} B^N) \times B^N \right) \\ &= \left(|B^N|^2 \operatorname{curl}^4 B^N - (B^N \cdot \nabla) \operatorname{curl}^3 B^N \times B^N - B^N \times (\operatorname{curl}^2 B^N \times B^N) \right) + R_2, \end{aligned}$$

by using (2.1), and we derive

$$\begin{aligned} J_5^3 &= -\gamma \int_{\mathcal{C}} |B^N|^2 |\operatorname{curl} B^N|^2 dx \\ &\quad + \gamma \int_{\mathcal{C}} |B^N \times \operatorname{curl} B^N|^2 dx \\ &\quad + \gamma \int_{\mathcal{C}} \operatorname{curl}^4 B^N \cdot ((B \cdot \nabla) \operatorname{curl}^3 B^N \times B^N) dx \\ &\quad - \gamma \int_{\mathcal{C}} \operatorname{curl}^4 B^N \cdot R_2 dx, \end{aligned} \quad (6.8)$$

where R_2 consists of the nonlinear terms for $D^k B^N$ for $0 \leq |k| \leq 3$. Unfortunately, it is not clear the sign of sum of the terms except for the last term on the right-hand side of (6.8).

However, in the case of S_5^m in \mathbb{R}^3 , we obtain

$$\begin{aligned} &S_5^m + \gamma \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} |(\operatorname{curl} D^\alpha B \times B)|^2 dx \\ &= -\gamma \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} (D^\alpha (B \times (\operatorname{curl} B \times B)) - (B \times (D^\alpha \operatorname{curl} B \times B))) \cdot (D^\alpha \operatorname{curl} B) dx. \end{aligned}$$

Here, the term $(D^\alpha (B \times (\operatorname{curl} B \times B)) - (B \times (D^\alpha \operatorname{curl} B \times B)))$ in the above has no m -th derivative of B , and

$$0 \leq \gamma \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} |(D^\alpha \operatorname{curl} B \times B)|^2 dx.$$

This yields

$$\left| S_5^m + \gamma \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} |(D^\alpha \operatorname{curl} B \times B)|^2 dx \right|$$

$$\lesssim \|DB(t, \cdot)\|_{H^m(\mathbb{R}^3)} \|B(t, \cdot)\|_{H^m(\mathbb{R}^3)}^3, \quad (6.9)$$

for $3 \leq m \in \mathbb{Z}$. Consequently, collecting (6.6), (6.7) and (6.9), one can deduce for any $3 \leq m \in \mathbb{Z}$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} (|D^\alpha u|^2 + |D^\alpha B|^2) dx + \sum_{0 \leq |\alpha| \leq m} \left(\int_{\mathbb{R}^3} \mu |D^{\alpha+1} u|^2 dx + \eta |D^{\alpha+1} B|^2 \right) dx \\ & + \gamma \sum_{0 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} |(D^\alpha \operatorname{curl} B \times B)|^2 dx \\ & \lesssim \left(\|u(t, \cdot)\|_{H^m(\mathbb{R}^3)}^{3/2} + \|B(t, \cdot)\|_{H^m(\mathbb{R}^3)}^{3/2} \right)^2 + \|DB(t, \cdot)\|_{H^m(\mathbb{R}^3)} \left(\|B(t, \cdot)\|_{H^m(\mathbb{R}^3)}^2 + \|B(t, \cdot)\|_{H^m(\mathbb{R}^3)}^3 \right). \end{aligned} \quad (6.10)$$

In the case $m = 2$, we can obtain similar higher-order energy estimates to Lemma 9, and the proofs of Theorem 2 (B) and Theorem 3 (B) in \mathbb{R}^3 can be shown. We skip the details, and introduce useful references [3, 6] for local well-posedness. Finally, using (6.10) for $3 \leq m \in \mathbb{Z}$, we can also prove Theorem 4 (B) in a similar fashion to the proof of Theorem 4 (A).

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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