Concentration and limit of large random matrices with given margins

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Based on joint work with Sumit Mukherjee

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Outline

Random matrix with i.i.d. entries given margins

- *(Base model)* $\mu =$ probability measure on $\mathbb{R}$, and let

$$A := \inf\{\mathrm{supp}(\mu)\} \leq \sup\{\mathrm{supp}(\mu)\} =: B.$$

$X \sim \mu^{\otimes(m \times n)}$: $(m \times n)$ random matrix with i.i.d. entries drawn from μ

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- *(Margins)* For a matrix $\mathbf{x} = (x_{ij}) \in \mathbb{R}^{m\times n}$, $(r(\mathbf{x}), c(\mathbf{x})) =$ margin of $\mathbf{x}$:

$$r(\mathbf{x}) := (r_1(\mathbf{x}), \dots, r_m(\mathbf{x})); \quad r_i(\mathbf{x}) := x_{i1} + \cdots + x_{in} \qquad (\triangleright \textit{ row margin of } \mathbf{x})$$
$$c(\mathbf{x}) = (c_1(\mathbf{x}), \dots, c_n(\mathbf{x})); \quad c_j(\mathbf{x}) := x_{1j} + \cdots + x_{mj} \qquad (\triangleright \textit{ column margin of } \mathbf{x})$$

For each $\rho \geq 0$, let

$$\mathcal{T}_\rho(\mathbf{r},\mathbf{c}) := \left\{ \mathbf{x} \in \mathbb{R}^{m\times n} \,:\, \|r(\mathbf{x}) - \mathbf{r}\|_1 \leq \rho,\ \|c(\mathbf{x}) - \mathbf{c}\|_1 \leq \rho \right\} \quad (\mathcal{T}(\mathbf{r},\mathbf{c}) := \mathcal{T}_0(\mathbf{r},\mathbf{c}))$$

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- *(Main question)* For ρ 'small',

If we condition $X \sim \mu^{\otimes(m \times n)}$ on being in $\mathcal{T}_\rho(\mathbf{r}, \mathbf{c})$, how does it look like?

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- This question still makes sense if μ is not a probability measure (i.e., counting measure on $\mathbb{N}$), as long as the measure of $\mathcal{T}_\rho(\mathbf{r}, \mathbf{c})$ under $\mu^{\otimes(m\times n)}$ is in $(0, \infty)$

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  - (*Maximum Liklihood Perspective*): The expectation of the maximum likelihood entrywise exponential tilting of the base model for margin $(\mathbf{r}, \mathbf{c})$

Exponential tilting

- $\mu_\theta :=$ exponentially tilted probability measure given by

$$\frac{d\mu_\theta}{d\mu}(x) = e^{\theta x - \psi(\theta)}, \quad \psi(\theta) := \log \int_{\mathbb{R}} e^{\theta x} d\mu(x) = \text{partition function}$$

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- For $\theta \in \Theta^\circ$, the **relative entropy** from the base mesure μ to the tilted probability measure μ_θ is

$$D(\mu_\theta \| \mu) := \int_{x \in \mathbb{R}} \log\left(\frac{d\mu_\theta}{d\mu}(x)\right) d\mu_\theta(x) = \theta\psi'(\theta) - \psi(\theta).$$

- Fix margins $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{R}^m$, $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$. The **typical table** Z for margin $(\mathbf{r}, \mathbf{c})$ is

$$Z^{\mathbf{r},\mathbf{c}} := \operatorname*{arg\,min}_{X \in \mathcal{T}(\mathbf{r},\mathbf{c}) \cap (A,B)^{m\times n}} \sum_{i,j} \underbrace{D(\mu_{\phi(x_{ij})} \,\|\, \mu)}_{D(\mu_{\phi(x)} \,\|\, \mu) = x\,\phi(x) - \psi(\phi(x))}$$

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- By multivariate Lagrange multipliers, there are 'dual variables' $\boldsymbol{\alpha} \in \mathbb{R}^m$, $\boldsymbol{\beta} \in \mathbb{R}^n$ s.t.

$$Z^{\mathbf{r},\mathbf{c}}_{ij} = \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j)) \quad \text{for all } i,j.$$

- (*Informal result I*)

  $X \sim \mu^{\otimes(m\times n)}$ **conditioned on being in** $\mathcal{T}_{\rho}(\mathbf{r},\mathbf{c})$ **concentrates around** $Z^{\mathbf{r},\mathbf{c}}$**,**
  where $Z^{\mathbf{r},\mathbf{c}}_{ij} = \psi'(\boldsymbol{\alpha}(i)+\boldsymbol{\beta}(j))$ **for some** $\boldsymbol{\alpha}\in\mathbb{R}^m, \boldsymbol{\beta}\in\mathbb{R}^n$

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- (*Informal result II*)

  $$\Big[\, X \sim \mu^{\otimes(m\times n)} \text{ conditioned on being in } \mathcal{T}_\rho(\mathbf{r},\mathbf{c}) \,\Big] \approx Y,$$

  where Y **has independent entries** $Y_{ij} \sim \mu_{\alpha(i)+\boldsymbol{\beta}(j)}$ **and** $\mathbb{E}[Y] = Z^{\mathbf{r},\mathbf{c}}$

Limit of a random matrix with i.i.d. entries given margins

- A **continuum margin** $(\mathbf{r}, \mathbf{c})$ = integrable functions $\mathbf{r}, \mathbf{c} : [0,1] \to \mathbb{R}$ such that $\int_0^1 \mathbf{r}(x)\, dx = \int_0^1 \mathbf{c}(y)\, dy$

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- For a vector $\mathbf{x} \in \mathbb{R}^m$, denote $\bar{\mathbf{x}}$ = the corresponding **step function** $[0,1] \to \mathbb{R}$,

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$$W_A(x,y) := A_{ij} \text{ if } (x,y) \in \left(\frac{i-1}{m}, \frac{i}{m}\right] \times \left(\frac{j-1}{n}, \frac{j}{n}\right]$$

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- (*Informal result III*)

  For $(\mathbf{r}_m, \mathbf{r}_n) \to (\mathbf{r}, \mathbf{c})$ in L^1 and $X \sim \mu^{\otimes(m\times n)}$ conditioned on $X \in \mathcal{T}_\rho(\mathbf{r}_m, \mathbf{c}_n)$, $W_X \to W^{\mathbf{r},\mathbf{c}}$ w.h.p. in 'cut norm'

  where $W^{\mathbf{r},\mathbf{c}}(x,y) = \psi'(\boldsymbol{\alpha}(x) + \boldsymbol{\beta}(y))$ for some $\boldsymbol{\alpha}, \boldsymbol{\beta} \in [0,1] \to \mathbb{R}$

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Relative entropy minimization given expected margins

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- (*Relative entropy*) For each probability measure $\mathcal{H}$ on $\mathbb{R}^{m\times n}$, the *relative entropy of $\mathcal{H}$ from $\mathcal{R}$* is

$$D_{KL}(\mathcal{H}\,\|\,\mathcal{R}) := \begin{cases} \int_{\mathbf{x}\in\mathbb{R}^{m\times n}} \log\left(\frac{d\mathcal{H}(\mathbf{x})}{d\mathcal{R}(\mathbf{x})}\right) \frac{d\mathcal{H}(\mathbf{x})}{d\mathcal{R}(\mathbf{x})}\,\mathcal{R}(d\mathbf{x}) & \textit{if } \mathcal{H}\ll\mathcal{R} \\ \infty & \text{otherwise.} \end{cases}$$

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- Find the least-action distribution $\mathcal{H}$ that minimizes the relative entropy from $\mathcal{R}$ constrained on the expected margins being $(\mathbf{r}, \mathbf{c})$:

$$\min_{\mathcal{H}\in\mathcal{P}^{m\times n}} D_{KL}(\mathcal{H}\,\|\,\mathcal{R}) \quad \text{subject to} \quad \mathbb{E}_{X\sim\mathcal{H}}[(r(X), c(X))] = (\mathbf{r}, \mathbf{c})$$

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  - Analogous to maximum entropy probability distributions given moments
  - e.g.,

$$\begin{aligned} \mathrm{Exp}(\lambda) &= \operatorname*{arg\,min}_{\mathcal{H}\in\mathcal{P}(\mathbb{R})} D_{KL}(\mathcal{H}\,|\,\mathrm{Leb}(\mathbb{R}_{\geq 0})) \quad \text{s.t.}\, \mathbb{E}[\mathcal{H}] = \lambda \\ &= \operatorname*{arg\,max}_{h} - \int_{\mathbb{R}} h(x)\log h(x)\, dx \quad \text{s.t.} \int_{\mathbb{R}} x\, h(x)\, dx = \lambda \end{aligned}$$

Relative entropy minimization given expected margins

- $(*)$ $$\min_{\mathcal{H}\in\mathcal{P}^{m\times n}} D_{KL}(\mathcal{H}\,\|\,\mathcal{R}) \quad \text{subject to} \quad \mathbb{E}_{X\sim\mathcal{H}}[(r(X), c(X))] = (\mathbf{r}, \mathbf{c}).$$

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- Specializing to $\mathcal{R}=\mu^{\otimes(m\times n)}$, the optimal measure $\mathcal{H}_{\text{opt}}$ is the product of some entrywise exponential tilting $\mu_{\theta_{ij}}$: $\mathcal{H}_{\text{opt}}=\bigotimes_{i,j}\mu_{\theta_{i,j}}$.

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  - Then the corresponding Lagrangian is

$$\mathcal{L}(h) = \int_{\mathbb{R}^{m\times n}} h(\mathbf{x})\log\left(h(\mathbf{x})\right)\, R(d\mathbf{x}) + \lambda\left(\int_{\mathbb{R}^{m\times n}} h(\mathbf{x})\, d\mathcal{R}(\mathbf{x}) - 1\right)$$

$$+\sum_{i=1}^{m}\alpha_i\left(\sum_{j=1}^{n}\int_{\mathbb{R}^{m\times n}} x_{ij}h(\mathbf{x})\, d\mathcal{R}(\mathbf{x}) - \mathbf{r}(i)\right) + \sum_{j=1}^{n}\beta_j\left(\sum_{i=1}^{m}\int_{\mathbb{R}^{m\times n}} x_{ij}h(\mathbf{x})\, d\mathcal{R}(\mathbf{x}) - \mathbf{c}(j)\right)$$

Relative entropy minimization given expected margins

- $(*)\qquad \min_{\mathcal{H}\in\mathcal{P}^{m\times n}} D_{KL}(\mathcal{H}\,\|\,\mathcal{R})$ subject to $\mathbb{E}_{X\sim\mathcal{H}}[(r(X),c(X))]=(\mathbf{r},\mathbf{c})$.
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  - Suppose the sol. of $(*)$ is attained in the interior. Then $\frac{\delta\mathcal{L}}{\delta h}=0$, so

$$\log h_{\text{opt}}(\mathbf{x})+1+\lambda+x_{ij}\alpha_i+x_{ij}\beta_j=0\quad\text{for all } i,j\quad \mathcal{R}\text{-a.s.}\ ,$$

$$\log h_{\text{opt}}(\mathbf{x})\propto\sum_{ij}(\alpha_i+\beta_{j,})x_{ij}\qquad \mathcal{R}\text{-a.s.}$$

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reduces to

$$\min_{Z\in\mathcal{T}(\mathbf{r},\mathbf{c})} \underbrace{D_{KL}\left(\bigotimes_{i,j}\mu_{\phi(z_{ij})} \,\middle\|\, \bigotimes_{i,j}\mu\right)}_{=\sum_{i,j} D(\mu_{\phi(z_{ij})}\,\|\,\mu)} \qquad (\triangleright \textit{ Typical table problem!})$$

- Specialze to $(m \times n) = (1 \times 2)$, 2-dim random vector $X = (X_1, X_2) \sim \mathcal{R}$. Constrain the marginal distributions, not the expected (row/column) margin:

$$\min_{\mathcal{H} \in \mathcal{P}^{1\times 2}} D_{KL}(\mathcal{H} \,\|\, \mathcal{R}) \quad \text{subject to} \quad X_1 =_d \mu_1,\, X_2 =_d \mu_2 \text{ where } (X_1, X_2) \sim \mathcal{H}.$$

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$$\frac{d\mathcal{H}}{d\mathcal{R}}(x, y) = e^{\boldsymbol{\alpha}_1(x) + \boldsymbol{\alpha}_2(y)} \qquad \mathcal{R}\text{-a.s.}$$

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- Schrödinger bridge $\leftarrow$ reweighting (from entrywise distributional constraint).

Outline

Introduction

Entropy minimization and Shrödinger bridge

Random graphs with given degree sequences

Contingency tables and Phase transition

Statement of results

Open problems

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- (*Question*)

How does a uniformly random graph with degree sequence d look like?

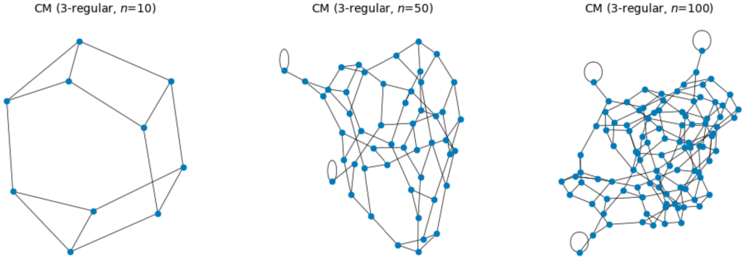

Figure: Random 3-regular graphs generated by the configuration model (allowing loops)

- $(\mathbf{d}^n)_{n\geq 1}$: dense degree sequence with scaling limit to $\mathbf{c} : [0,1] \to (c_1, c_2) \subseteq (0,1)$

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$$W^{\boldsymbol{\beta}^*}(x,y) = \frac{1}{1 + \exp(\boldsymbol{\beta}^*(x) + \boldsymbol{\beta}^*(y))}$$

    has 'degree sequence' $\mathbf{c}$:

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  - $A^n =$ Adj mx of the uniformly random graph with degree seq. $\mathbf{d}^n$. Then

$$W_{A^n} \to W^{\boldsymbol{\beta}^*} \qquad \textit{in weak cut distance},$$

  (W_{A^n}: step function corresponding to the adj mx A^n)

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Expected adjacency matrix:

$$\mathbb{E}[A^{\boldsymbol{\beta}}(i,j)] := \frac{e^{\boldsymbol{\beta}(i)+\boldsymbol{\beta}(j)}}{1 + e^{\boldsymbol{\beta}(i)+\boldsymbol{\beta}(j)}} = \psi'(\boldsymbol{\beta}(i) + \boldsymbol{\beta}(j)),$$

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  - (The MLE equation) $\frac{d\ell(\boldsymbol{\beta})}{d\boldsymbol{\beta}} = 0 \iff \mathbb{E}[\text{degree seq.}] = \mathbf{d}$:

$$\mathbb{E}\left[\sum_{j=1}^{n} A^{\boldsymbol{\beta}}(i,j)\right] = d_i \qquad \text{for all } 1 \leq i \leq n$$

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  - Putting things together:

$$W_{A^{\boldsymbol{\beta}_n}} \overset{\text{weak cut}}{\approx} W_{\mathbb{E}[A^{\boldsymbol{\beta}^n}]} = W^{\boldsymbol{\beta}^*} + o(1)$$

Outline

- Contingency tables = matrices with non-netative integer entries with fixed row an column margins

Data

1	0	3	2	0	7	13
1	2	0	4	3	0	10
7	5	2	1	0	0	15
0	0	3	1	3	9	16
0	3	1	8	0	2	14
5	3	0	3	5	3	19
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$v.s.$

Null model

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- Contingency tables are fundamental tools in statistics for studying dependence structure between two or more variables
- Uniform contingency table $X = (X_{ij})$ serves as the maximum entropy null model given margins

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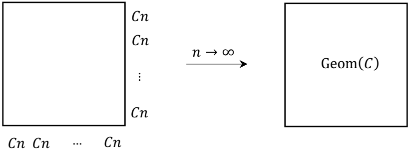

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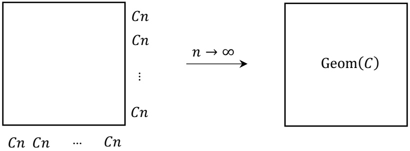
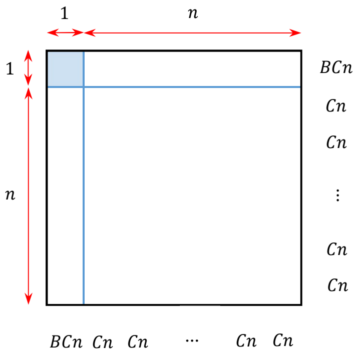


  - Empirical distribution of eigenvalues $\Rightarrow$ circular law (Nguyen '14 [12])

- What about non-uniform margins?



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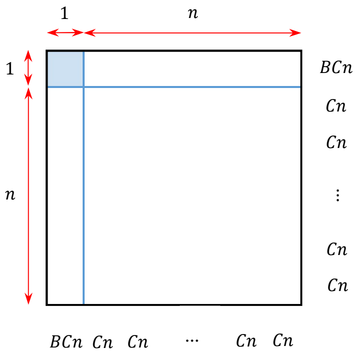

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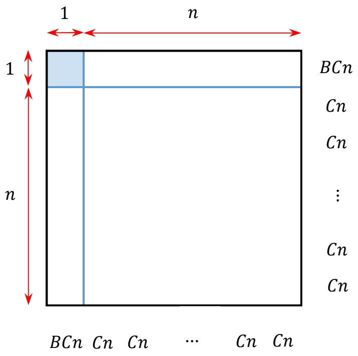

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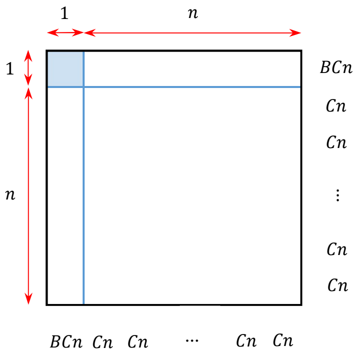

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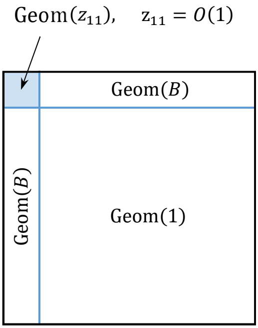
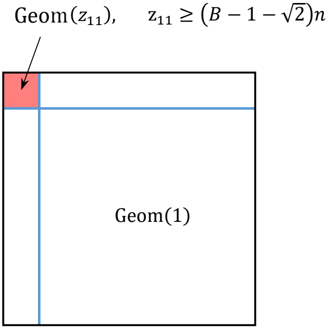
- Do we still have convergence to geometric entries for all $B, C \geq 1$?
- If so, what are the means of the geometric distribution in each block?

Barvinok's conjecture

- Based on a **typical table** computation, Barvinok conjectured in 2010 that each entry in X is asymptotically distributed as a geometric variable.



$B < 2$



$B > 1 + \sqrt{2} \approx 2.414$

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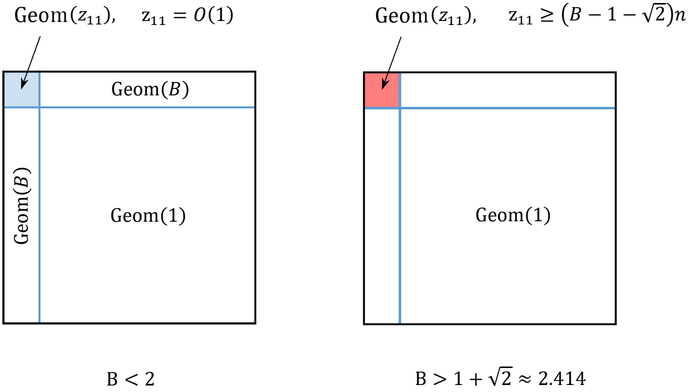
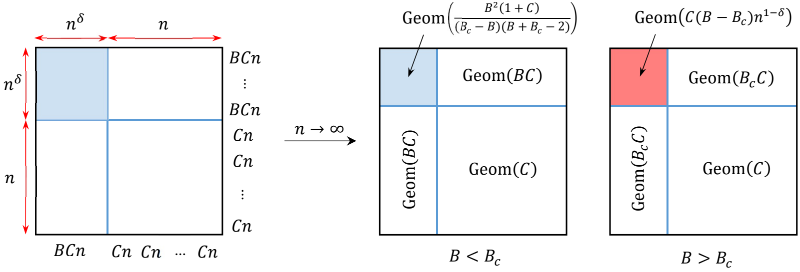
- Furthermore, for $C = 1$, he conjecture that $\mathbb{E}[X_{11}] = O(1)$ for $B < 2$ and $\mathbb{E}[X_{11}] = \Theta(n)$ for $B > 1 + \sqrt{2}$.



Theorem (Dittmer, L., and Pak '20)

Let $1/2 < \delta < 1$ and $\mathbf{a} = \mathbf{b} = (\overbrace{BCn, \ldots, BCn}^{n^\delta}, \overbrace{Cn, \ldots\ldots, Cn}^{n-n^\delta}) \in \mathbb{N}^n$. Let $B_c := 1 + \sqrt{1 + 1/C}$ and $X \sim \mathrm{Uniform}(\mathcal{T}(\mathbf{a}, \mathbf{b}))$. Then X marginally converges to the following matrix in total variation distance:



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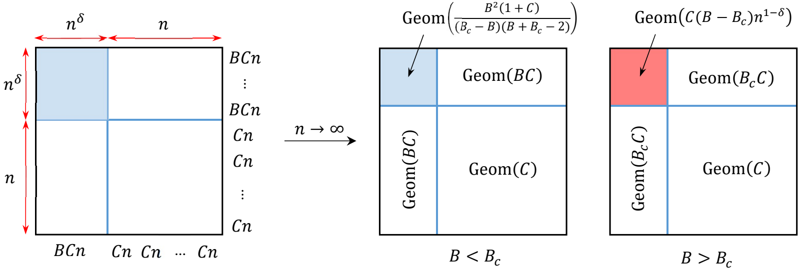

- **Where is this sharp phase transition coming from?**

- (Barvinok '10 [2]) For a $m \times n$ margin $(\mathbf{r}, \mathbf{c})$, the corresponding typical table is

$$Z^{\mathbf{r},\mathbf{c}} := \arg\max_{X \in \mathcal{T}(\mathbf{r},\mathbf{c})} \underbrace{\sum_{i,j} (x_{ij}+1)\log(x_{ij}+1) - x_{ij}\log(x_{ij})}_{=g(X)}$$

$$= \arg\max_{X \in \mathcal{T}(\mathbf{r},\mathbf{c})} \sum_{i,j} \text{Entropy}(\text{Geom}(\text{mean} = x_{ij}))$$

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  - Brändén, Leake, and Pak '23 [3] generalized this result to CTs with possibly bounded integer values (Using Lorenzian polynomials)

Sharp phase transition in typical tables

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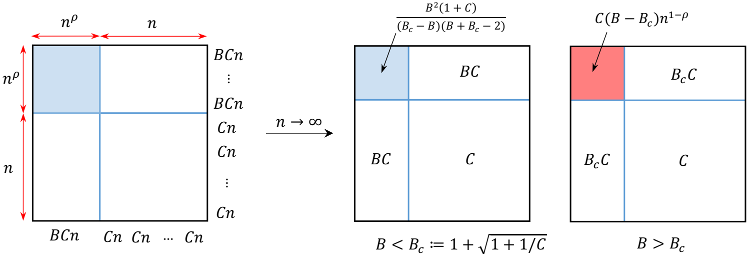

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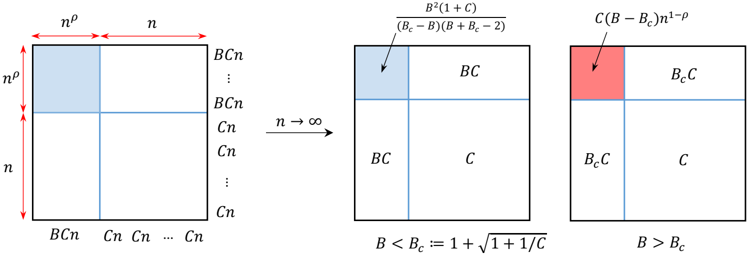
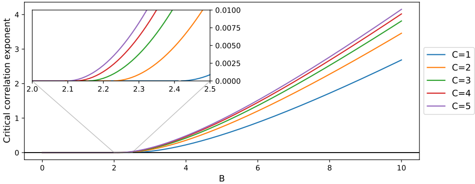


  - This result was used to obtain a second-order phase transition in the number of CTs with Barvinok margin by Lyu and Pak '22 [10]



Asmptotic independence $\xrightarrow{B \nearrow}$ Positive correlation

Outline

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  - In particular, for $\mu = \text{Counting}(\mathbb{N})$, then

$$\begin{aligned} \delta\text{-tame} \quad &\Longleftarrow \quad t/s < \rho_c := 1+\sqrt{1+s^{-1}} \\ \text{Diverging typical tables} \quad &\Longleftarrow \quad t/s > \rho_c \end{aligned}$$

  (Generalizes the subcritical behavior of Barvinok margin) ($\rho_c = B_c$ for $s = C$)

Theorem (L., and Muhkerjee '24+)

$(\mathbf{r},\mathbf{c}) = (m \times n)$ *δ-tame margin,* $Z = Z^{\mathbf{r},\mathbf{c}}$ *Let* $X = (X_{ij}) \sim \mu^{\otimes(m\times n)}$*. Then for some constants* $C_i = C_i(\delta,\mu) > 0$ *for* $i = 1, 2$*,*

(i) $\exists$ *MLE* $\boldsymbol{\alpha} \in \mathbb{R}^m$*,* $\boldsymbol{\beta} \in \mathbb{R}^n$ *such that* $Z_{ij} = \psi'(\boldsymbol{\alpha}(i) + \boldsymbol{\beta}(j))$ *for all* i, j*.*

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(iii) *Denote* $s(m,n) := mn\sqrt{m^{-1} + n^{-1}}$. *Then*

$$\mathbb{P}\left(\|W_X - W_Z\|_\square \geq C_1 \left(m^{-1} + n^{-1}\right)^{1/4} \,\middle|\, X \in \mathcal{T}_{C_1 s(m,n)}(\mathbf{r}, \mathbf{c}) \right) \leq \exp\left(-C_2 s(m,n)\right).$$

Theorem (L., and Muhkerjee '24+)

$(\mathbf{r}_m, \mathbf{c}_n) =$ *seq. of* $m \times n$ δ*-tame margins* $\to$ *continuum margin* $(\mathbf{r}, \mathbf{c})$ *in* L^1 *as* $m, n \to \infty$.

(i) $\exists$ *bounded measurable* $\boldsymbol{\alpha}, \boldsymbol{\beta} : [0,1] \to \mathbb{R}$ *such that the kernel*

$$W^{\mathbf{r},\mathbf{c}}(x, y) := \psi'(\boldsymbol{\alpha}(x) + \boldsymbol{\beta}(y))$$

has continuum margin $(\mathbf{r}, \mathbf{c})$.

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$(\mathbf{r}_m, \mathbf{c}_n)$ = seq. of $m \times n$ δ-tame margins $\to$ continuum margin $(\mathbf{r}, \mathbf{c})$ in L^1 as $m, n \to \infty$.

(i) *$\exists$ bounded measurable $\boldsymbol{\alpha}, \boldsymbol{\beta} : [0,1] \to \mathbb{R}$ such that the kernel*

$$W^{\mathbf{r},\mathbf{c}}(x,y) := \psi'(\boldsymbol{\alpha}(x) + \boldsymbol{\beta}(y))$$

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(ii) *For $C = C(\mu, \delta) > 0$,*

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(iii) *Denote* $s(m,n) := mn\sqrt{m^{-1} + n^{-1}}$ *and let* $X \sim \mu^{\otimes(m \times n)}$ *be conditional on* $X \in \mathcal{T}_{s(m,n)}(\mathbf{r}_m, \mathbf{c}_n)$. *Then with prob. at least* $1 - \exp(-C_1 s(m,n))$,

$$\| W_X - W^{\mathbf{r},\mathbf{c}} \|_\square \leq \underbrace{C_2 \left(m^{-1} + n^{-1} \right)^{1/4}}_{\text{fluctuation around } W_{Z^{\mathbf{r}_m,\mathbf{c}_n}}} + \underbrace{C_2 \sqrt{\| (\mathbf{r}, \mathbf{c}) - (\bar{\mathbf{r}}_m, \bar{\mathbf{c}}_n) \|}}_{\text{bias } (*)}$$

Corollary (L., and Muhkerjee '24+)

Assume $\mathrm{Supp}(\mu) = \{0, 1, \cdots, B\}$ *for some* $B \in \{1, 2, \cdots, \infty\}$.

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- δ-tame condition $\Leftarrow$ 'asymmetric EG-condition' for **(i)**, 'subcriticality' for **(ii)**

Outline

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    (*Natural matrix-loss function γ? Spectral norm?*)

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Thank you very much!

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